

The decomposability of smash product of \mathbf{A}_n^2 -complexes

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Abstract In this paper, we determine the decomposability of smash product of two indecomposable \mathbf{A}_n^2 -complexes, i.e., $(n-1)$ -connected finite CW-complexes with dimension at most $n+2$ ($n \geq 3$).

keywords indecomposable; smash product; \mathbf{A}_n^k -complexes; cofibre sequence.

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1 Introduction

For a finite CW-complex X with co-H-space structure, if $X \simeq X_1 \vee X_2$ with non-contractible X_1 and X_2 , then X is called decomposable; otherwise X is called indecomposable. One of the basic problems in homotopy theory is to classify indecomposable homotopy types. Although, it is impossible to find all indecomposable homotopy types, it is indeed possible to solve this problem in special situations. Let \mathbf{A}_n^k ($n \geq k+1$), the homotopy category consisting of $(n-1)$ -connected finite CW-complexes with dimension at most $n+k$, any complex in \mathbf{A}_n^k is a suspension and thus a co-H-space; \mathbf{F}_n^k , the full subcategory of \mathbf{A}_n^k consisting of complexes with torsion free homology groups; $\mathbf{F}_{n(2)}^k$, the full subcategories of \mathbf{A}_n^k consisting of complexes with 2-torsion free homology groups; $\mathbf{F}_{n(2,3)}^k$, the full subcategories of \mathbf{A}_n^k consisting of complexes with 2 and 3 torsion free homology groups.

In 1950, S.C. Chang classified the indecomposable homotopy types in \mathbf{A}_n^2 ($n \geq 3$) [6], that is

- (i) Spheres: S^n, S^{n+1}, S^{n+2} ;

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- (ii) Elementary Moore spaces: $M_{p^r}^n, M_{p^r}^{n+1}$ where p is a prime, $r \in \mathbb{Z}^+$ and $M_{p^r}^k$ denotes $M(\mathbb{Z}/p^r, n)$;
- (iii) Elementary Chang complexes: $C_\eta^{n+2}, C^{(n+2),s}, C_r^{(n+2)}, C_r^{(n+2),s}$ (See Section 2.1) where $r, s \in \mathbb{Z}^+$.

where \mathbb{Z}^+ denotes the set of positive integers.

Classification of indecomposable homotopy types of $\mathbf{A}_n^3 (n \geq 4)$ are given by Baues and Hennes in [5] and for $k \geq 4$, classification of indecomposable homotopy types of $\mathbf{A}_n^k (n \geq k+1)$ is wild in the sense similar to that in representation theory of finite dimensional algebras. Classification of indecomposable homotopy types of $\mathbf{F}_n^k (n \geq k+1)$ for $k = 4, 5, 6$ are given by Baues and Drozd in [3], [4] and [8] and it is wild for $k \geq 7$. Based on these earlier results, in our previous papers [12] and [13], we classify indecomposable homotopy types of $\mathbf{F}_{n(2,3)}^k (n \geq k+1)$ for $k \leq 6$ and $\mathbf{F}_{n(2)}^4 (n \geq 5)$.

As pointed out by Jie Wu [18], starting from an explicit space X , one obtains more indecomposable spaces from the self-smash products of X since self-smash products of co-H-spaces admit decompositions. This motivates us to consider the decomposability of smash products of different indecomposable complexes. There are only a few results for this problem. The decomposability of $M(\mathbb{Z}/p^r, n) \wedge M(\mathbb{Z}/p^s, n)$ is well known [10]. Jie Wu [17] proved that $M(\mathbb{Z}/2, n) \wedge C_\eta^{n+2}$ and $C_\eta^{n+2} \wedge C_\eta^{n+2}$ are indecomposable. As a main result in this paper, we determine the decomposability of all remaining smash product of two indecomposable complexes in $\mathbf{A}_n^2 (n \geq 3)$ and give the decomposition whenever possible. Since the suspension functor $\Sigma : \mathbf{A}_n^2 \rightarrow \mathbf{A}_{n+1}^2$ is an equivalence for $n \geq 3$. It suffices to deal with the case $n = 3$.

Theorem 1.1 (Main theorem). *For $r, s, r', s', u \in \mathbb{Z}^+$,*

- $C_r^5 \wedge C^{5,s}, C_r^5 \wedge C_{r'}^5, C^{5,s} \wedge C^{5,s'}, C_\eta^5 \wedge C_r^5, C_\eta^5 \wedge C^{5,s}, C_\eta^5 \wedge C_r^{5,s}$ are indecomposable;
- $M_{2^u}^3 \wedge C_\eta^5$ is indecomposable;
- $M_{2^u}^3 \wedge C_r^5$ is
 - ◊ indecomposable for $u > r$;
 - ◊ homotopy equivalent to $M_{2^u}^3 \wedge C_\eta^5 \vee M_{2^u}^7$ for $r \geq u$;
- $M_{2^u}^3 \wedge C^{5,s}$ is
 - ◊ indecomposable for $u > s$;
 - ◊ homotopy equivalent to $M_{2^u}^3 \wedge C_\eta^5 \vee M_{2^u}^7$ for $s \geq u$
- $M_{2^u}^3 \wedge C_r^{5,s}$ is homotopy equivalent to
 - ◊ $C_r^{8,s} \vee C_r^{9,s}$ for $u > r, s$;

- ◇ $M_{2u}^3 \wedge C_r^5 \vee M_{2u}^7$ for $r < u \leq s$;
 - ◇ $M_{2u}^3 \wedge C^{5,s} \vee M_{2u}^7$ for $s < u \leq r$;
 - ◇ $M_{2u}^3 \wedge C_\eta^5 \vee M_{2u}^7 \vee M_{2u}^7$ for $u \leq r$ and $u \leq s$;
- $C_u^5 \wedge C_r^{5,s}$ is
 - ◇ homotopy equivalent to $C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s}$ for $u \geq r$ and $u \geq s$;
 - ◇ homotopy equivalent to $C_s^{9,r} \vee C_\eta^5 \wedge C_s^{5,s}$ for $u = s < r$;
 - ◇ indecomposable, otherwise;
- $C^{5,u} \wedge C_r^{5,s}$ is
 - ◇ homotopy equivalent to $C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s}$ for $u \geq r$ and $u \geq s$;
 - ◇ homotopy equivalent to $C_s^{9,r} \vee C_\eta^5 \wedge C_r^{5,r}$ for $u = r < s$;
 - ◇ indecomposable, otherwise;
- $C_r^{5,s} \wedge C_{r'}^{5,s'}$
 - ◇ if $s \geq r, r', s'$

$$C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq \begin{cases} C_r^{9,s} \vee C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s}, & s = s' = r' > r; \\ C_r^{9,s} \vee C_{r'}^{5,s'} \wedge C_r^{5,s}, & s = s' > r' > r; \\ C_r^{9,s} \vee C_{r'}^{5,s'} \wedge C_{r'}^{5,s}, & s = r' > s' > r; \\ C_{r'}^{9,s'} \vee C_{r'}^{9,s'} \vee C_\eta^5 \wedge C_{r'}^{5,s'}, & s \geq r \geq r', s'; \\ C_{r'}^{9,s'} \vee C_{s'}^{9,r'} \vee C_\eta^5 \wedge C_r^{5,r}, & s \geq r' > s' = r; \\ C_{r'}^{9,s'} \vee C_r^{5,s} \wedge C_{r'}^{5,s'}, & \text{otherwise.} \end{cases}$$

- ◇ if $r \geq r', s'$ and $r > s$

$$C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq \begin{cases} C_r^{9,s} \vee C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s}, & r = r' = s' > s; \\ C_r^{9,s} \vee C_{r'}^{5,s'} \wedge C_r^{5,s}, & r = r' > s' > s; \\ C_r^{9,s} \vee C_{r'}^{5,s'} \wedge C_r^{5,s}, & r = s' > r' > s; \\ C_{r'}^{9,s'} \vee C_{r'}^{9,s'} \vee C_\eta^5 \wedge C_{r'}^{5,s'}, & r > s \geq r', s'; \\ C_{r'}^{9,s'} \vee C_{s'}^{9,r'} \vee C_\eta^5 \wedge C_s^{5,s}, & r \geq s' > r' = s; \\ C_{r'}^{9,s'} \vee C_r^{5,s} \wedge C_{r'}^{5,s'}, & \text{otherwise.} \end{cases}$$

Remark 1.2. For $M_{p^r}^3$, prime $p \neq 2$, it is easy to check that $M_{p^r}^3 \wedge C_\eta^5 \simeq M_{p^r}^6 \vee M_{p^r}^8$; $M_{p^r}^3 \wedge C_r^{5,s} \simeq M_{p^r}^8$; $M_{p^r}^3 \wedge C^{5,s} \simeq M_{p^r}^6$; $M_{p^r}^3 \wedge C_r^{5,s} \simeq *$; $M_{p^r}^3 \wedge M_{q^r}^3 \simeq *$ for prime $q \neq p$ ($*$ is the point space). We will not discuss these cases any more in the following.

It is known from Theorem 1.2 of [15] that for any p -local CW-complex, there is a functorial decomposition $\Omega \Sigma X \simeq A^{\min}(X) \times \Omega(\bigvee_{n=2}^{\infty} Q_n^{\max}(X))$ which is useful to calculate homotopy groups of ΣX . $Q_n^{\max}(X)$ is a wedge summand of $\Sigma X^{(n)}$, where

$X^{(n)}$ is a n -fold self-smash product of X . In order to determine the homotopy type of $Q_n^{max}(X)$, it is significant to decompose $X^{(n)}$ to a wedge of indecomposable spaces. The decomposition of self-smash product is easy for $M_{p^r}^n(p > 2)$ and $M_{2^s}^n(s > 1)$. The decomposition of self-smash product is obtained by Jie Wu [17] for M_2^n and C_η^n . In a sequel we will study the decomposition of self-smash product for Chang complexes.

Main Method (Assume C_1 and C_2 are indecomposable homotopy types in \mathbf{A}_3^2)

The indecomposability of $C_1 \wedge C_2$ is obtained by contradiction. Assuming that $C_1 \wedge C_2$ is decomposable one gets a contradiction by computing its homotopy invariants such as homotopy groups, cohomotopy groups or Steenrod operations.

There are two ways to get the wedge decomposition of $C_1 \wedge C_2$:

One way: Applying Lemma 2.3 to cofibre sequence $X \xrightarrow{f} Y \rightarrow C_1$ to get $X \wedge C_2 \xrightarrow{f \wedge 1} Y \wedge C_2 \rightarrow C_1 \wedge C_2$ which is also a cofibre sequence. Then rewrite $f \wedge 1 \simeq (f_1, f_2, \dots, f_t)$ under identification $X \wedge C_2 \simeq X_1 \vee X_2 \vee \dots \vee X_t$ or rewrite $f \wedge 1 \simeq \begin{pmatrix} f'_1 \\ f'_2 \\ \dots \\ f'_{t'} \end{pmatrix}$ under identification $Y \wedge C_2 \simeq Y_1 \vee Y_2 \vee \dots \vee Y_{t'}$ and prove that $f_i \simeq 0$ for some i or $f'_j \simeq 0$ for some j which will imply that ΣX_i or Y_j is a wedge summand of $C_1 \wedge C_2$.

Another way: Firstly, observe that $C_1 \wedge C_2$ is a CW-complex with only one top cell e^{10} and one bottom cell S^6 ; cancel the top cell and pinch the bottom cell to a point to get spaces $(C_1 \wedge C_2)^{(9)}$ and $(C_1 \wedge C_2)/S^6$ respectively. The two spaces have mapping cone structures by Lemma 2.4; Secondly, decompose $(C_1 \wedge C_2)^{(9)} \simeq U_1 \vee U_2 \vee \dots \vee U_l$ and $(C_1 \wedge C_2)/S^6 \simeq W_1 \vee W_2 \vee \dots \vee W_m$ by matrix techniques introduced briefly in Subsection 2.1. At last, from the decomposition of $(C_1 \wedge C_2)^{(9)}$, there is a cofibre sequence $S^9 \xrightarrow{f} U_1 \vee U_2 \vee \dots \vee U_l \rightarrow C_1 \wedge C_2$ and the map f is determined by the decomposition of $(C_1 \wedge C_2)/S^6$.

Section 2 contains necessary notations and lemmas. Related results of elementary Moore spaces and Chang-complexes are stated in Section 3. In Section 5, we prove the last part of Theorem 1.1 by determining the decomposition of $C_r^{5,s} \wedge C_{r'}^{5,s'}$ while the proof of other cases in Theorem 1.1 is given in Section 4.

2 Preliminaries

2.1 Some notations

- All spaces are suspensions of simply connected finite CW-complexes.
- $|G|$ denotes the order of a group G and $|g|$ denotes the order of an element g in group G . If G is an abelian group with decomposition $G \cong C_1 \oplus C_2 \oplus \dots \oplus C_m$,

where C_t is a cyclic group with order infinity or a power of a prime for $t = 1, \dots, m$, then define $\dim G := m$.

- If X is a subspace of L , $Y \simeq L/X$, then i denotes the canonical inclusion $X \hookrightarrow L$, q denotes the canonical projection $L \twoheadrightarrow Y$. Especially for Moore space $M_{2^k}^n$, sometimes we denote $i : S^n \hookrightarrow M_{2^k}^n$ by i_n and $q : M_{2^k}^n \twoheadrightarrow S^{n+1}$ by q_n .
- Denote by $H_*X := H_*(X; \mathbb{Z})$ and $H^*(X; \mathbb{Z}/2)$ the **reduced** homology groups and cohomology groups of space X respectively.
- Let \mathbf{C}_f be the mapping cone of a map f . Denote by $[\mathbf{C}_f, \mathbf{C}_{f'}]_\beta^\alpha$ the set of homotopy classes of maps h which satisfy the the following homotopy commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & \mathbf{C}_f & \longrightarrow & \Sigma X \\ \downarrow \alpha & & \downarrow \beta & & \downarrow h & & \downarrow \Sigma \alpha \\ X' & \xrightarrow{f'} & Y' & \longrightarrow & \mathbf{C}_{f'} & \longrightarrow & \Sigma X' \end{array}$$

- For abelian groups A_i and B_j , ($i = 1, \dots, t, j = 1, \dots, s$), we denote by $f := (f_{ij}) = \begin{pmatrix} f_{11} & \cdots & f_{1t} \\ \cdots & \cdots & \cdots \\ f_{s1} & \cdots & f_{st} \end{pmatrix} : \bigoplus_{i=1}^t A_i \rightarrow \bigoplus_{j=1}^s B_j$ a morphism such that $p_{B_i} f j_{A_j} = f_{ij}$, where j_{A_j} and p_{B_i} are canonical inclusions and projections respectively. Sometimes, (f_{ij}) is written graphically as follows to indicate the domain and codomain

$$\begin{array}{c} A_1 \quad A_2 \quad \cdots \quad A_t \\ B_1 \quad \boxed{\begin{array}{cccc} f_{11} & f_{12} & \cdots & f_{1t} \\ f_{21} & f_{22} & \cdots & f_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ f_{s1} & f_{s2} & \cdots & f_{st} \end{array}} \\ B_2 \\ \vdots \\ B_s \end{array}$$

- For finite CW-complexes X_i and Y_j , ($i = 1, \dots, t, j = 1, \dots, s$), let

$$f := (f_{ij}) = \begin{pmatrix} f_{11} & \cdots & f_{1t} \\ \cdots & \cdots & \cdots \\ f_{s1} & \cdots & f_{st} \end{pmatrix} : \bigvee_{i=1}^t X_i \rightarrow \bigvee_{j=1}^s Y_j$$

be a map such that $p_{Y_i} f j_{X_j} \simeq f_{ij}$, where j_{X_j} and p_{Y_i} are canonical inclusions and projections respectively. Similarly, (f_{ij}) is written as

$$\begin{array}{c} X_1 \quad X_2 \quad \cdots \quad X_t \\ Y_1 \quad \boxed{\begin{array}{cccc} f_{11} & f_{12} & \cdots & f_{1t} \\ f_{21} & f_{22} & \cdots & f_{2t} \\ \vdots & \vdots & \vdots & \vdots \\ f_{s1} & f_{s2} & \cdots & f_{st} \end{array}} \\ Y_2 \\ \vdots \\ Y_s \end{array}$$

Generally, (f_{ij}) is not unique up to homotopy. However, if $s = 1$ or X_i ($i = 1, \dots, t$) and Y_j ($j = 1, \dots, s$) are in the category \mathbf{A}_n^k for some $k \geq n+2$, then (f_{ij}) is unique.

Thus in category $\mathbf{A}_n^k (k \geq n+2)$,

$$[\bigvee_{i=1}^t X_i, \bigvee_{j=1}^s Y_j] \cong \{(f_{ij}) | f_{ij} \in [X_i, Y_j]\}$$

The composition (sum) of the maps is compatible with the product (sum) of the matrices. Thus a self-map is homotopy equivalent if and only if the corresponding matrix is invertible. Moreover for two matrices

$$(f_{ij}), (g_{ij}) : \bigvee_{i=1}^t X_i \rightarrow \bigvee_{j=1}^s Y_j,$$

we call $(f_{ij}) \cong (g_{ij})$ if there are invertible matrices

$$(\alpha_{ij}) : \bigvee_{i=1}^t X_i \rightarrow \bigvee_{i=1}^t X_i, \quad (\beta_{ij}) : \bigvee_{j=1}^s Y_j \rightarrow \bigvee_{j=1}^s Y_j$$

such that $(\beta_{ij})(f_{ij})(\alpha_{ij}) \simeq (g_{ij})$. It is clear that if $(f_{ij}) \cong (g_{ij})$, then the mapping cones $\mathbf{C}_{(f_{ij})}$ and $\mathbf{C}_{(g_{ij})}$ are homotopy equivalent to each other.

To simplify the text, we fix names for some elementary transformations as follows

- (i) $-\mathbf{r}_n$ ($-\mathbf{c}_n$) : composing n -th row (column) with $-id$;
- (ii) $\mathbf{c}_m f + \mathbf{c}_n$: adding the m -th column, composed with map f , to the n -th column;
- (iii) $g \mathbf{r}_m + \mathbf{r}_n$: adding the m -th row, composed with map g , to the n -th row;
- (iv) $k \mathbf{r}_m + \mathbf{r}_n$ ($k \mathbf{c}_m + \mathbf{c}_n$): adding k times of the m -th row (column) to the n -th row (column), where $k \in \mathbb{Z}^+$;
- In the following of the paper, $S_\omega^k = S^k$ for $\omega \in \{1, 2, 3, 4, a, b\}$.
- $\eta = \eta_n \in [S^{n+1}, S^n]$ is a Hopf map for $n \geq 3$ and $\varrho = \varrho_n \in [S^{n+3}, S^n] \cong \mathbb{Z}/24$ is a fixed generator for $n \geq 5$.
- $\kappa, \kappa', \varepsilon, \varepsilon' \in \{0, 1\}$.
- The k times of the identity map id of ΣX is written as $k : \Sigma X \rightarrow \Sigma X$ for nonzero integer k (hence, $1 = id$);

- For $k \geq 5$, and $r, s \in \mathbb{Z}^+$, let

$$\begin{aligned}
C^{k,s} &= (S^{k-2} \vee S^{k-1}) \cup \binom{\eta}{2^s} \mathbf{C}S^{k-1} = S^{k-2} \cup_{\eta q} \mathbf{C}M_{2^s}^{k-2}; \\
C_r^k &= S^{k-2} \cup_{(2^r, \eta)} \mathbf{C}(S^{k-2} \vee S^{k-1}) = M_{2^r}^{k-2} \cup_{i\eta} \mathbf{C}S^{k-1}; \\
C_r^{k,s} &= (S^{k-2} \vee S^{k-1}) \cup \binom{2^r, \eta}{0, 2^s} \mathbf{C}(S^{k-2} \vee S^{k-1}) = (M_{2^r}^{k-2} \vee S^{k-2}) \cup \binom{i\eta}{2^s} \mathbf{C}S^{k-1} \\
&= S^{k-2} \cup_{(2^r, \eta q)} \mathbf{C}(S^{k-2} \vee M_{2^s}^{k-2}) = M_{2^r}^{k-1} \cup_{i\eta q} \mathbf{C}M_{2^s}^{k-1}; \\
C_\eta^k &= S^{k-2} \cup_\eta \mathbf{C}S^{k-1},
\end{aligned}$$

2.2 Spanier-Whitehead duality

If \mathbf{A}_n^k is in the stable range, i.e., $[X, Y] \xrightarrow{\Sigma^m} [\Sigma^m X, \Sigma^m Y]$ is isomorphic for any $X, Y \in \mathbf{A}_n^k$ and any $m \in \mathbb{Z}^+$, then there is a contravariant isomorphism of additive categories

$$D = D_{2n+k} : \mathbf{A}_n^k \rightarrow \mathbf{A}_n^k$$

which is called Spanier-Whitehead duality (or $(2n+k)$ -duality).

D satisfies the following properties from [2], [7] and [14]:

Proposition 2.1.

- (i) D^2 is equal to the identity functor;
- (ii) $[X, Y] \xrightarrow{D(\cong)} [DY, DX] \cong [DY \wedge X, S^{2n+k}];$
- (iii) $[S^{n+q}, DX] \cong [X, S^{n+k-q}]$ and $[S^{n+q}, X] \cong [DX, S^{n+k-q}]$ for $n \leq q \leq n+k$;
- (iv) $D(X \vee Y) \simeq DX \vee DY$;
- (v) $D(X \wedge Y) \simeq DX \wedge DY$, that is for $X \in \mathbf{A}_n^k, Y \in \mathbf{A}_m^l$, then $X \wedge Y \in \mathbf{A}_{n+m}^{k+l}$ and $D_{2(n+m)+k+l}(X \wedge Y) \simeq D_{2n+k}X \wedge D_{2m+l}Y$.
- (vi) Let $\{X, Y\} := \lim_{m \rightarrow +\infty} [\Sigma^m X, \Sigma^m Y]$. Then for any CW-complex Z ,

$$\{X \wedge Y, Z\} \cong \{X \wedge S^{2n+k}, DY \wedge Z\}$$

Note. It follows from (i) and (iv) above that X is indecomposable if and only if DX is indecomposable.

Example 2.2. (Page 49 of [1]) For the Spanier-Whitehead duality $D : \mathbf{A}_n^2 \rightarrow \mathbf{A}_n^2$ ($n \geq 3$), we have $DS^n = S^{n+2}$, $DS^{n+1} = S^{n+1}$, $DM_{p^r}^n = M_{p^r}^{n+1}$, $DC_\eta^{n+2} = C_\eta^{n+2}$, $DC_r^{(n+2),s} = C_s^{(n+2),r}$, $DC_r^{n+2} = C^{(n+2),r}$.

2.3 Some lemmas

Lemma 2.3. ([7]) For a cofibre sequence $X \xrightarrow{f} Y \xrightarrow{i} \mathbf{C}_f$,

$$X \wedge Z \xrightarrow{f \wedge id} Y \wedge Z \xrightarrow{i \wedge id} \mathbf{C}_f \wedge Z$$

is also a cofibre sequence. That is $\mathbf{C}_{f \wedge id} \simeq \mathbf{C}_f \wedge Z$.

Lemma 2.4. (Lemma 14.30. of [14]) For

$$X \xrightarrow{f} U \xrightarrow{i} \mathbf{C}_f, \quad Y \xrightarrow{g} V \xrightarrow{i} \mathbf{C}_g$$

$\mathbf{C}_f \wedge \mathbf{C}_g = (U \wedge V) \bigcup_{\mu} \mathbf{C}(X \wedge V \vee U \wedge Y) \bigcup_{\nu} \mathbf{C}\mathbf{C}(X \wedge Y)$ where $\mu = (f \wedge id, id \wedge g)$ and $(\mathbf{C}_f \wedge \mathbf{C}_g)/(U \wedge V) \simeq (\Sigma(X \wedge V) \vee \Sigma(U \wedge Y)) \bigcup_{\nu'} \mathbf{C}\Sigma(X \wedge Y)$, where $\nu' = \begin{pmatrix} \Sigma id \wedge g \\ -\Sigma f \wedge id \end{pmatrix}$.

Lemma 2.5. (Lemma 6.2.1 of [11]) Let

$$X \xrightarrow{f} Y \xrightarrow{i} \mathbf{C}_f \xrightarrow{q} \Sigma X$$

be a cofibre sequence. If f is null homotopic, then

- (i) there is a retraction $r : \mathbf{C}_f \rightarrow Y$ of i , such that $ri \simeq id$ and $\mathbf{C}_f \rightarrow \Sigma X \vee \mathbf{C}_f \xrightarrow{1 \vee r} \Sigma X \vee Y$ is a homotopy equivalence, where the first map is the standard coaction.
- (ii) there is a section $\tau : \Sigma X \rightarrow \mathbf{C}_f$ such that $q\tau \simeq id$ and $\Sigma X \vee Y \xrightarrow{(\tau, i)} \mathbf{C}_f$ is a homotopy equivalence.

Lemma 2.6. Let $A \in \mathbf{A}_6^4$, with homology groups of the form $\mathbb{Z}^r \oplus \mathbb{Z}/2^{r_1} \oplus \cdots \oplus \mathbb{Z}/2^{r_s}$ for some nonnegative integers r, r_1, \dots, r_s . Suppose that

- (i) $\dim H_9 A + \dim H_{10} A = 1$ and $H^6(A; \mathbb{Z}/2) \cong H^{10}(A; \mathbb{Z}/2) \cong \mathbb{Z}/2$ with generators a_6 and a_{10} respectively, satisfying $Sq^4 a_6 = a_{10}$;
- (ii) $\dim H^8(A; \mathbb{Z}/2) \geq 2$ and there are nonzero elements $a_8 \neq a'_8 \in H^8(A; \mathbb{Z}/2)$ such that $Sq^2 a_8 = Sq^2 a'_8 = a_{10}$;
- (iii) Moreover $Sq^2 a_6 = a_8 + a'_8 + a''_8 \neq 0$ for some $a''_8 \in H^8(A; \mathbb{Z}/2)$ such that $Sq^2 a''_8 = 0$.

If $A \simeq X \vee Y$ and $H_6 X \neq 0$, then $H_t X \cong H_t A$ for $t = 6, 9, 10$ and $\dim H^8(X; \mathbb{Z}/2) \geq 2$, hence $\dim H_7 X + \dim H_8 X \geq 2$.

Proof. Let

$$X \begin{matrix} \xrightarrow{j_1} \\ \xleftarrow{p_1} \end{matrix} A \begin{matrix} \xleftarrow{j_2} \\ \xrightarrow{p_2} \end{matrix} Y$$

where j_1, j_2, p_1, p_2 be the canonical inclusions and projections.

$$H^*(X; \mathbb{Z}/2) \begin{matrix} \xrightarrow{p_1^*} \\ \xleftarrow{j_1^*} \end{matrix} H^*(A; \mathbb{Z}/2) \begin{matrix} \xleftarrow{p_2^*} \\ \xrightarrow{j_2^*} \end{matrix} H^*(Y; \mathbb{Z}/2),$$

where $j_u^* p_u^* = id$ which implies that p_u^* is injective and j_u^* is surjective for $u = 1, 2$.

Since $H_6 X \neq 0$, we get that $H_6 X \cong H_6 A$, $H_6 Y = 0$ and $H^6(p_1; \mathbb{Z}/2)$ is isomorphic, hence there is $0 \neq x_6 \in H^6(X; \mathbb{Z}/2)$ such that $p_1^*(x_6) = a_6$. It follows from $p_1^*(Sq^4 x_6) = Sq^4 p_1^*(x_6) = a_{10} \neq 0$ that $0 \neq Sq^4 x_6 \in H^{10}(X; \mathbb{Z}/2)$ which implies $H_9 X \cong H_9 A$ and $H_{10} X \cong H_{10} A$ by (i). By $j_1^*(a_{10}) = j_1^*(p_1^*(Sq^4 x_6)) = Sq^4 x_6 \neq 0$ and $Sq^2 j_1^*(a_8) = Sq^2 j_1^*(a'_8) = j_1^*(a_{10})$, we get $j_1^*(a_8) \neq 0$ and $j_1^*(a'_8) \neq 0$ in $H^8(X; \mathbb{Z}/2)$.

Since $Sq^2 a_6 = a_8 + a'_8 + a''_8 \neq 0$ and $Sq^2 a''_8 = 0$, $p_1^*(Sq^2 x_6) = Sq^2(a_6) = a_8 + a'_8 + a''_8 \neq 0$ and $p_1^* Sq^2 Sq^2 x_6 = 2a_{10} = 0$, thus $Sq^2 x_6 \neq 0$ and $Sq^2 Sq^2 x_6 = 0$. But $Sq^2(j_1^*(a_8)) = j_1^*(a_{10}) \neq 0$, we have $j_1^*(a_8) \neq Sq^2 x_6$, thus $\dim H^8(X; \mathbb{Z}/2) \geq 2$.

It follows from $H^8(X; \mathbb{Z}/2) = Hom(H_8 X, \mathbb{Z}/2) \oplus Ext(H_7 X, \mathbb{Z}/2)$ that $\dim H_7 X + \dim H_8 X \geq 2$. \square

A complex X is called 2-local if all homotopy groups or equivalently all homology groups of X are finitely generated $\mathbb{Z}_{(2)}$ -module, where $\mathbb{Z}_{(2)}$ is 2-localization of \mathbb{Z} . Let $X_{(2)}$ be the 2-localization of X and denote by $X \simeq_{(2)} Y$ if $X_{(2)} \simeq Y_{(2)}$.

Lemma 2.7.

- (i) Let $X_1 = S^m \cup_{f_1} \mathbf{C}X'_1$, $X_2 = S^m \cup_{f_2} \mathbf{C}X'_2$ be two (resp. 2-local) complexes, where X'_1 and X'_2 are m -connected. If $X_1 \simeq X_2$ and $H_m X_1 = \mathbb{Z}/2^s$ for some $s \in \mathbb{Z}^+$, then $X_1/S^m \simeq_{(2)} X_2/S^m$ (resp. $X_1/S^m \simeq X_2/S^m$), i.e., $\Sigma X'_1 \simeq_{(2)} \Sigma X'_2$ (resp. $\Sigma X'_1 \simeq \Sigma X'_2$).
- (ii) Let $X_1 = X_1^{(n-1)} \cup_{g_1} \mathbf{C}S^{n-1}$, $X_2 = X_2^{(n-1)} \cup_{g_2} \mathbf{C}S^{n-1}$ be two (resp. 2-local) complexes, where $X_1^{(n-1)}$ and $X_2^{(n-1)}$ are $(n-1)$ -skeleton of X_1 and X_2 respectively. If $X_1 \simeq X_2$ and $[X_1, S^n] = \mathbb{Z}/2^t$ for some $t \in \mathbb{Z}^+$, then $X_1^{(n-1)} \simeq_{(2)} X_2^{(n-1)}$ (resp. $X_1^{(n-1)} \simeq X_2^{(n-1)}$).

Proof. It suffices to prove when X_1 and X_2 are 2-local.

The proof of (i):

There are cofibre sequences

$$X'_j \xrightarrow{f_j} S^m \xrightarrow{i_j} X_j \xrightarrow{q_j} \Sigma X'_j \xrightarrow{\Sigma f_j} S^{m+1} \quad j = 1, 2.$$

Given a homotopy equivalence $\alpha : X_1 \xrightarrow{\simeq} X_2$, it induces

$$\alpha_* : \pi_m X_1 \xrightarrow{\cong} \pi_m X_2 \cong \mathbb{Z}/2^s.$$

Since i_j is a generator of $\pi_m X_j$ for $j = 1, 2$, there is an odd integer k such that $\alpha_* k i_1 = i_2 \in \pi_m X_2$. Since X_1 and X_2 are 2-local, $k\alpha : X_1 \rightarrow X_2$ is also a homotopy equivalence. Thus there is a commutative diagram

$$\begin{array}{ccccccc} S^m & \xrightarrow{i_1} & X_1 & \xrightarrow{q_1} & \Sigma X'_1 & \longrightarrow & S^{m+1} \\ \parallel & & \downarrow k\alpha(\simeq) & & \downarrow \alpha' & & \parallel \\ S^m & \xrightarrow{i_2} & X_2 & \xrightarrow{q_2} & \Sigma X'_2 & \longrightarrow & S^{m+1} \end{array}$$

and

$$\alpha' : \Sigma X'_1 \xrightarrow{\simeq} \Sigma X'_2.$$

The proof of (ii) is dual to the proof of (i) by investigating the cofibre sequences

$$S^{n-1} \xrightarrow{g_j} X_j^{(n-1)} \xrightarrow{i_j} X_j \xrightarrow{q_j} S^n \quad j = 1, 2.$$

and using isomorphism $[X_1, S^n] \cong [X_2, S^n]$. \square

3 Moore spaces and Chang-complexes

In this section, we will collect some basic facts about Chang-complexes and Moore spaces.

3.1 Moore spaces

Firstly, from Proposition 3E.3 of [9], the Steenrod square action on $M_{2^r}^n$ is given by

$$Sq^1 : H^n(M_{2^r}^n; \mathbb{Z}/2) \rightarrow H^{n+1}(M_{2^r}^n; \mathbb{Z}/2) \text{ is } \begin{cases} \text{isomorphic, } r = 1; \\ 0, & r > 1. \end{cases}$$

Secondly, we list some results of maps between Moore spaces from [5].

$$[M_{2^r}^n, M_{2^t}^n] = \begin{cases} \mathbb{Z}/4\langle B(\chi) \rangle, & r = t = 1; \\ \mathbb{Z}/2^{\min(r,t)}\langle B(\chi) \rangle \oplus \mathbb{Z}/2\langle i\eta q \rangle, & \text{ohterwise.} \end{cases} \quad (n \geq 3)$$

where $B(\chi)$ is given by Proposition (2.3) of [5], which satisfies

$$H_n B(\chi) = \chi : \mathbb{Z}/2^r \rightarrow \mathbb{Z}/2^t, \quad \chi(1) = 1.$$

$B(\chi) \in [M_{2^r}^n, M_{2^t}^n]_{id}^{2^{r-t}}$ for $r \geq t$ and $B(\chi) \in [M_{2^r}^n, M_{2^t}^n]_{2^{r-t}}^{id}$ for $r \leq t$; If $r = t$, then $B(\chi) = id$ and if $r = t = 1$, then $i\eta q = 2B(\chi) = 2$.

$$[S^{n+1}, M_{2^t}^n] = \mathbb{Z}/2\langle i\eta \rangle, \quad [M_{2^t}^n, S^n] = \mathbb{Z}/2\langle \eta q \rangle \quad (n \geq 3)$$

$$[M_{2^t}^{n+1}, S^n] = \begin{cases} \mathbb{Z}/4\langle \eta^1 \rangle, & t = 1; \\ \mathbb{Z}/2\langle \eta^t \rangle \oplus \mathbb{Z}/2\langle \eta\eta q \rangle, & t > 1. \end{cases} \quad (n \geq 3)$$

$$[S^{n+2}, M_{2^t}^n] = \begin{cases} \mathbb{Z}/4\langle \xi_1 \rangle, & t = 1; \\ \mathbb{Z}/2\langle \xi_t \rangle \oplus \mathbb{Z}/2\langle i\eta\eta \rangle, & t > 1. \end{cases} \quad (n \geq 4)$$

Here we choose a generator ξ_1 and set $\xi_t = B(\chi)\xi_1$. The generator η^1 is the dual map of ξ_1 and $\eta^t = \eta^1 B(\chi)$. And $q\xi_t = \eta$, $\eta^t i = \eta$ for $t \geq 1$.

$$[M_{2^s}^{n+1}, M_{2^r}^n] = \begin{cases} \mathbb{Z}/2\langle \xi_1^1 \rangle \oplus \mathbb{Z}/2\langle \eta_1^1 \rangle, & s = r = 1; \\ \mathbb{Z}/4\langle \xi_1^s \rangle \oplus \mathbb{Z}/2\langle \eta_1^s \rangle, & s > 1 = r; \\ \mathbb{Z}/2\langle \xi_r^1 \rangle \oplus \mathbb{Z}/4\langle \eta_r^1 \rangle, & s = 1 < r; \\ \mathbb{Z}/2\langle \xi_r^s \rangle \oplus \mathbb{Z}/2\langle \eta_r^s \rangle \oplus \mathbb{Z}/2\langle i\eta\eta q \rangle, & \text{otherwise,} \end{cases} \quad (n \geq 4)$$

where $\xi_r^s = B(\chi)\xi_1 \in [M_{2^s}^{n+1}, M_{2^r}^n]_0^\eta$, $\eta_r^s = i\eta^1 B(\chi) \in [M_{2^s}^{n+1}, M_{2^r}^n]_\eta^0$. Note that

$$2\xi_1^s = i\eta\eta q \ (s > 1); \quad 2\eta_r^1 = i\eta\eta q \ (r > 1).$$

Let $\lambda_{11} := \xi_r^s + \eta_r^s$, then

$$[M_{2^s}^{n+1}, M_{2^r}^n]_\eta^\eta = \{\lambda_{11}, \lambda_{11} + i\eta\eta q\}.$$

$[S^{n+3}, M_{2^r}^n]$ is given by the following Lemma

Lemma 3.1. *Let $n \geq 5$. Then*

$$[S^{n+3}, M_2^n] = \mathbb{Z}/2\langle i\varrho \rangle \oplus \mathbb{Z}/2\langle \rho_1 \rangle$$

$$[S^{n+3}, M_{2^r}^n] = \mathbb{Z}/4\langle i\varrho \rangle \oplus \mathbb{Z}/2\langle \rho_r \rangle \quad (r > 1)$$

where ρ_r is some element of $[S^{n+3}, M_{2^r}^n]$ such that $q\rho_r = \eta\eta$ for $r \geq 1$.

Proof. $[S^{n+3}, M_2^n]$ is obtained from Lemma 5.2 and Theorem 5.11 of [17]. For $r > 1$ there are two exact sequences

$$[S^k, S^n] \xrightarrow{(2^r)^*} [S^k, S^n] \rightarrow [S^k, M_{2^r}^n] \xrightarrow{q^*} [S^k, S^{n+1}] \xrightarrow{(2^r)^*} [S^k, S^{n+1}], \quad k = n+2, n+3.$$

The following commutative diagram is induced by $\eta : S^{n+3} \rightarrow S^{n+2}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & [S^{n+2}, M_{2^r}^n] & \xrightleftharpoons[\sigma]{q^*} & \mathbb{Z}/2 \longrightarrow 0 \\ & & \downarrow & & \downarrow \eta^* & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & [S^{n+3}, M_{2^r}^n] & \xrightarrow{q^*} & \mathbb{Z}/2 \longrightarrow 0 \end{array}$$

It is known that the upper exact sequence splits. If σ is the section of q^* , then $\eta^* \sigma$ is the section of the lower q^* . \square

At last, it follows from Lemma 2.5 that

Corollary 3.2. For $s \geq r$ and $s > 1$

$$M_{2^r}^n \wedge S^m \xrightleftharpoons[\tau_m]{1 \wedge i_m} M_{2^r}^n \wedge M_{2^s}^m \xrightleftharpoons[\sigma_m]{1 \wedge q_m} M_{2^r}^n \wedge S^{m+1}$$

and

$$S^m \wedge M_{2^r}^n \xrightleftharpoons[\tau'_m]{i_m \wedge 1} M_{2^s}^m \wedge M_{2^r}^n \xrightleftharpoons[\sigma'_m]{q_m \wedge 1} S^{m+1} \wedge M_{2^r}^n$$

where $\tau_m(1 \wedge i_m) = 1$, $(1 \wedge q_m)\sigma_m = 1$ and $\tau'_m(i_m \wedge 1) = 1$, $(q_m \wedge 1)\sigma'_m = 1$.

3.2 Chang-complexes

Firstly, since $C_1^{k,1}$ and $\Sigma^{k-4}M_2^1 \wedge M_2^1$ are both indecomposable \mathbf{A}_{k-2}^2 -complexes with the same homology groups, thus

$$C_1^{k,1} \simeq \Sigma^{k-4}M_2^1 \wedge M_2^1 \quad (k \geq 5).$$

Cofibre sequences for Chang-complexes

For $C \in \{C_r^k, C_r^{k,s}, C_r^{k,s} \mid k \geq 5, r, s \in \mathbb{Z}^+\}$, it can be written as mapping cones of different maps, that is, there are different cofibre sequences for C .

- The cofibre sequence for C_η^k

$$\mathbf{Cof} : S^{k-1} \xrightarrow{\eta} S^{k-2} \xrightarrow{i_\eta} C_\eta^k \xrightarrow{q_\eta} S^k \rightarrow S^{k-1}$$

- The cofibre sequences for C_r^k

$$\mathbf{Cof1} : S^{k-2} \vee S^{k-1} \xrightarrow{(2^r, \eta)} S^{k-2} \xrightarrow{i_S} C_r^k \xrightarrow{q_S} S^{k-1} \vee S^k \rightarrow S^{k-1};$$

$$\mathbf{Cof2} : S^{k-1} \xrightarrow{i_\eta} M_{2^r}^{k-2} \xrightarrow{i_M} C_r^k \xrightarrow{q_M} S^k \rightarrow M_{2^r}^{k-1};$$

$$\mathbf{Cof3} : S^{k-2} \xrightarrow{i_\eta 2^r} C_\eta^k \xrightarrow{i_C} C_r^k \xrightarrow{q_C} S^{k-1} \rightarrow C_\eta^{k+1};$$

- The cofibre sequences for $C_r^{k,s}$

$$\mathbf{Cof1} : S^{k-1} \xrightarrow{\begin{pmatrix} \eta \\ 2^s \end{pmatrix}} S^{k-2} \vee S^{k-1} \xrightarrow{i_S} C_r^{k,s} \xrightarrow{q_S} S^k \rightarrow S^{k-1} \vee S^k;$$

$$\mathbf{Cof2} : M_{2^s}^{k-2} \xrightarrow{\eta q} S^{k-2} \xrightarrow{i_M} C_r^{k,s} \xrightarrow{q_M} M_{2^s}^{k-1} \rightarrow S^{k-1};$$

$$\mathbf{Cof3} : C_\eta^{k-1} \xrightarrow{2^s q_\eta} S^{k-1} \xrightarrow{i_C} C_r^{k,s} \xrightarrow{q_C} C_\eta^k \rightarrow S^k;$$

- The cofibre sequences for $C_r^{k,s}$

$$\mathbf{Cof1} : S^{k-2} \vee S^{k-1} \xrightarrow{\begin{pmatrix} 2^r, \eta \\ 0, 2^s \end{pmatrix}} S^{k-2} \vee S^{k-1} \xrightarrow{i_S} C_r^{k,s} \xrightarrow{q_S} S^{k-1} \vee S^k \rightarrow S^{k-1} \vee S^k;$$

$$\begin{aligned}
\mathbf{Cof2} : M_{2^s}^{k-2} &\xrightarrow{i\eta q} M_{2^r}^{k-2} \xrightarrow{i_M} C_r^{k,s} \xrightarrow{q_M} M_{2^s}^{k-1} \rightarrow M_{2^r}^{k-1}; \\
\mathbf{Cof3} : S^{k-2} \vee M_{2^s}^{k-2} &\xrightarrow{(2^r, \eta q)} S^{k-2} \xrightarrow{i_M} C_r^{k,s} \xrightarrow{q_M} S^{k-1} \vee M_{2^s}^{k-1} \rightarrow S^{k-1}; \\
\mathbf{Cof4} : S^{k-1} &\xrightarrow{\begin{pmatrix} i\eta \\ 2^s \end{pmatrix}} M_{2^r}^{k-2} \vee S^{k-1} \xrightarrow{i_M} C_r^{k,s} \xrightarrow{q_M} S^k \rightarrow M_{2^r}^{k-1} \vee S^k; \\
\mathbf{Cof5} : C_r^{k-1} &\xrightarrow{2^s p_1 q_S} S^{k-1} \xrightarrow{i_C} C_r^{k,s} \xrightarrow{q_C} C_r^k \rightarrow S^k, \text{ where } 2^s p_1 q_S \text{ is the composi-} \\
&\text{tion of } C_r^{k-1} \xrightarrow{q_S} S^{k-1} \vee S^{k-2} \xrightarrow{p_1} S^{k-1} \xrightarrow{2^s} S^{k-1}; \\
\mathbf{Cof6} : S^{k-2} &\xrightarrow{i_S j_1 2^r} C_r^{k,s} \xrightarrow{i_{\overline{C}}} C_r^{k,s} \xrightarrow{q_{\overline{C}}} S^{k-1} \rightarrow C^{(k+1),s}, \text{ where } i_S j_1 2^r \text{ is the} \\
&\text{composition of } S^{k-2} \xrightarrow{2^r} S^{k-2} \xrightarrow{j_1} S^{k-2} \vee S^{k-1} \xrightarrow{i_S} C_r^{k,s}.
\end{aligned}$$

Homologies and Cohomologies

$$\begin{aligned}
H_* C_r^{k,s} &= \begin{cases} \mathbb{Z}, & * = k-2; \\ \mathbb{Z}/2^s, & * = k-1; \\ 0, & \text{otherwise.} \end{cases} & H_* C_r^k &= \begin{cases} \mathbb{Z}/2^r, & * = k-2; \\ \mathbb{Z}, & * = k; \\ 0, & \text{otherwise.} \end{cases} \\
H_* C_r^{k,s} &= \begin{cases} \mathbb{Z}/2^r, & * = k-2; \\ \mathbb{Z}/2^s, & * = k-1; \\ 0, & \text{otherwise.} \end{cases} & H_* C_\eta^k &= \begin{cases} \mathbb{Z}, & * = k-2; \\ \mathbb{Z}, & * = k; \\ 0, & \text{otherwise.} \end{cases} \\
H^*(C_r^k; \mathbb{Z}/2) &= H^*(C_r^{k,s}; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & * = k-2, k-1, k; \\ 0, & \text{otherwise.} \end{cases} \\
H^*(C_r^{k,s}; \mathbb{Z}/2) &= \begin{cases} \mathbb{Z}/2, & * = k-2, k; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & * = k-1; \\ 0, & \text{otherwise.} \end{cases} & H^*(C_\eta^k; \mathbb{Z}/2) &= \begin{cases} \mathbb{Z}/2, & * = k-2; \\ \mathbb{Z}/2, & * = k; \\ 0, & \text{otherwise.} \end{cases} \\
Sq^2 : H^{k-2}(C; \mathbb{Z}/2) &\rightarrow H^k(C; \mathbb{Z}/2) \text{ is isomorphic for } C = C_r^{k,s}, C_r^k, C_r^{k,s}, C_\eta^k. \\
Sq^1 = 0 : H^{k-2}(C_r^{k,s}; \mathbb{Z}/2) &\rightarrow H^{k-1}(C_r^{k,s}; \mathbb{Z}/2) \\
Sq^1 : H^{k-1}(C_r^{k,s}; \mathbb{Z}/2) &\rightarrow H^k(C_r^{k,s}; \mathbb{Z}/2) \text{ is } \begin{cases} 0, & s > 1; \\ \text{isomorphic}, & s = 1. \end{cases} \\
Sq^1 : H^{k-2}(C_r^k; \mathbb{Z}/2) &\rightarrow H^{k-1}(C_r^k; \mathbb{Z}/2) \text{ is } \begin{cases} 0, & r > 1; \\ \text{isomorphic}, & r = 1. \end{cases} \\
Sq^1 = 0 : H^{k-1}(C_r^k; \mathbb{Z}/2) &\rightarrow H^k(C_r^k; \mathbb{Z}/2). \\
Sq^1 \text{ on } H^*(C_r^{k,s}; \mathbb{Z}/2) &\text{ is given by the following Lemma}
\end{aligned}$$

Lemma 3.3. *Let v_{k-2}, v_k be generators of $H^{k-2}(C_r^{k,s}; \mathbb{Z}/2)$ and $H^k(C_r^{k,s}; \mathbb{Z}/2)$ respectively. Then there are generators v_{k-1}, \bar{v}_{k-1} of $H^{k-1}(C_r^{k,s}; \mathbb{Z}/2)$ such that*

$$Sq^1 v_{k-2} = \begin{cases} v_{k-1}, & r = 1; \\ 0, & r > 1. \end{cases}; \quad Sq^1 \bar{v}_{k-1} = \begin{cases} v_k, & s = 1; \\ 0, & s > 1. \end{cases}; \quad Sq^1 v_{k-1} = 0. \quad (1)$$

Proof. To simplify the notation, take $k = 5$. Using **Cof2** and **Cof3** of $C_r^{5,s}$, we have a split exact sequence

$$0 \rightarrow H^4(M_{2^s}^4; \mathbb{Z}/2) \xrightarrow{q_M^*} H^4(C_r^{5,s}; \mathbb{Z}/2) \xrightarrow{i_M^*} H^4(M_{2^r}^3; \mathbb{Z}/2) \rightarrow 0$$

and an isomorphism

$$H^4(S^4; \mathbb{Z}/2) \oplus H^4(M_{2^s}^4; \mathbb{Z}/2) \xrightarrow{(p_1^*, p_2^*) \cong} H^4(S^4 \vee M_{2^s}^4; \mathbb{Z}/2) \xrightarrow{q_M^* \cong} H^4(C_r^{5,s}; \mathbb{Z}/2)$$

where p_l is the canonical projection for $l = 1, 2$. Let u_S^4, u_M^4 be the generators of $H^4(S^4; \mathbb{Z}/2)$ and $H^4(M_{2^s}^4; \mathbb{Z}/2)$ respectively.

Note that $p_2 q_M = q_M$. Define

$$\bar{v}_4 := q_M^* p_2^*(u_M^4) = q_M^*(u_M^4), \quad v_4 := \begin{cases} Sq^1 v_3, & r = 1; \\ q_M^* p_1^*(u_S^4), & r > 1. \end{cases}$$

Clearly, $v_4 \neq \bar{v}_4$ for $r > 1$. For $r = 1$, since $H^3(C_1^{5,s}; \mathbb{Z}/2) \xrightarrow{i_M^* (\cong)} H^3(M_2^3; \mathbb{Z}/2)$, $i_M^* v_4 = i_M^* Sq^1 v_3 = Sq^1 i_M^* v_3 \neq 0$. But $i_M^* \bar{v}_4 = 0$, we also get $v_4 \neq \bar{v}_4$.

$$Sq^1 \bar{v}_4 = Sq^1 q_M^* u_M^4 = q_M^* Sq^1 u_M^4 \begin{cases} v_5, & s = 1; \\ 0, & s > 1. \end{cases}$$

$$Sq^1 v_4 = \begin{cases} Sq^1 q_M^* p_1^* u_S^4 = q_M^* p_1^* Sq^1 u_S^4 = 0, & r > 1; \\ Sq^1 Sq^1 v_3 = 0, & r = 1. \end{cases}$$

Applying **Cof4** of $C_r^{5,s}$ and by the commutative diagram

$$\begin{array}{ccc} H^3(C_r^{5,s}; \mathbb{Z}/2) & \xrightarrow{i_M^*} & H^3(M_{2^r}^3 \vee S^4; \mathbb{Z}/2) \\ Sq^1 \downarrow & & \downarrow Sq^1 \\ H^4(C_r^{5,s}; \mathbb{Z}/2) & \xrightarrow{i_M^*} & H^4(M_{2^r}^3 \vee S^4; \mathbb{Z}/2) \end{array}$$

we get

$$Sq^1 v_3 = \begin{cases} v_4 \neq 0, & r = 1; \\ 0, & r > 1. \end{cases}$$

□

homotopy groups and cohomotopy groups

For $k \geq 5$

$$\pi_{k-1} C_r^k = 0;$$

$$\pi_k C_r^k = \mathbb{Z}/2 \langle (j_1 \eta)_S^- \rangle \oplus \mathbb{Z} \langle (2j_2)_S^- \rangle \text{ where } (j_1 \eta)_S^- = q_{S*}^{-1}(j_1 \eta) \text{ and } (2j_2)_S^- = q_{S*}^{-1}(2j_2);$$

$$\pi_{k-1} C^{k,s} = \mathbb{Z}/2^{s+1} \langle i_S j_2 \rangle \text{ with } i_S j_1 \eta = 2^s i_S j_2;$$

$$\pi_k C^{k,s} = \mathbb{Z}/2 \langle i_S j_2 \eta \rangle;$$

$$\pi_{k-1} C_r^{k,s} = \mathbb{Z}/2^{s+1} \langle i_M j_2 \rangle \text{ with } i_M j_1 i \eta = 2^s i_M j_2 \text{ or}$$

$$\pi_{k-1} C_r^{k,s} = \mathbb{Z}/2^{s+1} \langle i_S j_2 \rangle \text{ with } i_S j_1 i \eta = 2^s i_S j_2;$$

$$\begin{aligned}
\pi_k C_r^{k,s} &= \mathbb{Z}/2\langle i_{\underline{M}} j_1 \xi_r \rangle \oplus \mathbb{Z}/2\langle i_{\underline{M}} j_2 \eta \rangle; \\
[C_r^k, S^{k-2}] &= \mathbb{Z}/2\langle \eta p_1 q_S \rangle; \\
[C_r^k, S^{k-1}] &= \mathbb{Z}/2^{r+1}\langle p_1 q_S \rangle \text{ with } \eta p_2 q_S = 2^r p_1 q_S; \\
[C_r^{k,s}, S^{k-2}] &= \mathbb{Z}\langle (2p_1)_{\bar{S}} \rangle \oplus \mathbb{Z}/2\langle (\eta p_2)_{\bar{S}} \rangle \text{ where } (2p_1)_{\bar{S}} = (i_S^*)^{-1}(2p_1) \text{ and } (\eta p_2)_{\bar{S}} = (i_S^*)^{-1}(\eta p_2); \\
[C_r^{k,s}, S^{k-1}] &= 0; \\
[C_r^{k,s}, S^{k-2}] &= \mathbb{Z}/2\langle (\eta q p_1)_{\bar{M}} \rangle \oplus \mathbb{Z}/2\langle (\eta p_2)_{\bar{M}} \rangle \text{ where } (\eta q p_1)_{\bar{M}} = (i_{\underline{M}}^*)^{-1}(\eta q p_1) \\
&\text{and } (\eta p_2)_{\bar{M}} = (i_{\underline{M}}^*)^{-1}(\eta p_2); \\
[C_r^{k,s}, S^{k-1}] &= \mathbb{Z}/2^{r+1}\langle p_1 q_S \rangle \text{ with } \eta p_2 q_S = 2^r p_1 q_S \text{ or} \\
[C_r^{k,s}, S^{k-1}] &= \mathbb{Z}/2^{r+1}\langle p_1 q_{\bar{M}} \rangle \text{ with } \eta q p_2 q_{\bar{M}} = 2^r p_1 q_{\bar{M}}; \\
[C_r^{k,s}, S^k] &= \mathbb{Z}/2^s\langle q p_2 q_{\underline{M}} \rangle = \mathbb{Z}/2^s\langle p_1 q_S \rangle.
\end{aligned}$$

where $X_u \xrightarrow{j_u} X_1 \vee X_2$ is the canonical inclusion and $X_1 \vee X_2 \xrightarrow{p_u} X_u$ is the canonical projections for $u = 1, 2$.

Remark 3.4. $[X, Y]$ for X, Y being indecomposable homotopy types of \mathbf{A}_n^2 are given by Part IV of [1]. We will use these results directly in the following of this paper.

4 Determination of the decomposability except that of $C_r^{5,s} \wedge C_{r'}^{5,s'}$

4.1 Some indecomposable cases

Theorem 4.1. $M_{2^u}^3 \wedge C_\eta^5$, $C_r^5 \wedge C_\eta^{5,s}$, $C_r^5 \wedge C_{r'}^5$, $C_\eta^{5,s} \wedge C_{r'}^{5,s'}$, $C_\eta^5 \wedge C_r^5$, $C_\eta^5 \wedge C_\eta^{5,s}$ and $C_\eta^5 \wedge C_r^{5,s}$ are indecomposable for any $u, r, r', s, s' \in \mathbb{Z}^+$.

Proof. The proof of all cases in the Theorem 4.1 are similar. We give a proof only for the case $C_r^5 \wedge C_{r'}^5$.

Let u_k, u'_k be generators of $H^k(C_r^5; \mathbb{Z}/2)$ and $H^k(C_{r'}^5; \mathbb{Z}/2)$ respectively for $k = 3, 4, 5$. Let m_{uv} be the minimum of non-negative integers u and v .

$$H_*(C_r^5 \wedge C_{r'}^5) = \begin{cases} \mathbb{Z}/2^{m_{r,r'}}, & * = 6 \\ \mathbb{Z}/2^{m_{r,r'}}, & * = 7 \\ \mathbb{Z}/2^r \oplus \mathbb{Z}/2^{r'}, & * = 8 \\ 0, & * = 9 \\ \mathbb{Z}, & * = 10 \\ 0, & \text{otherwise} \end{cases}$$

$$H^*(C_r^5 \wedge C_{r'}^5; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\{u_3 \otimes u'_3\}, & * = 6 \\ \mathbb{Z}/2\{u_3 \otimes u'_4, u_4 \otimes u'_3\}, & * = 7 \\ \mathbb{Z}/2\{u_3 \otimes u'_5, u_4 \otimes u'_4, u_5 \otimes u'_3\}, & * = 8 \\ \mathbb{Z}/2\{u_4 \otimes u'_5, u_5 \otimes u'_4\}, & * = 9 \\ \mathbb{Z}/2\{u_5 \otimes u'_5\}, & * = 10 \\ 0, & \text{otherwise} \end{cases}$$

By the Steenrod operation action on Chang-complexes given in Section 3.2, we get

- (i) $Sq^4(u_3 \otimes u'_3) = u_5 \otimes u'_5$; (ii) $Sq^2(u_3 \otimes u'_5) = Sq^2(u_5 \otimes u'_3) = u_5 \otimes u'_5$;
- (iii) $Sq^2(u_3 \otimes u'_3) = \begin{cases} u_3 \otimes u'_5 + u_5 \otimes u'_3, & \text{otherwise} \\ u_3 \otimes u'_5 + u_4 \otimes u'_4 + u_5 \otimes u'_3, & r = r' = 1 \end{cases}$, $Sq^2(u_4 \otimes u'_4) = 0$;
- (iv) $Sq^2(u_3 \otimes u'_4) = u_5 \otimes u'_4$, $Sq^2(u_4 \otimes u'_3) = u_4 \otimes u'_5$.

Suppose that $C_r^5 \wedge C_{r'}^5 = X \vee Y$ and $H_6 X \neq 0$, $C_r^5 \wedge C_{r'}^5$ satisfies the conditions in Lemma 2.6,

$$H_t X \cong H_t(C_r^5 \wedge C_{r'}^5), \quad t = 6, 9, 10; \quad \dim H_7 X + \dim H_8 X \geq 2,$$

which implies that $\sum_{t=1}^{\infty} \dim H_t Y \leq 1$.

It follows from the isomorphism $Sq^2 : H^7(C_r^5 \wedge C_{r'}^5; \mathbb{Z}/2) \rightarrow H^9(C_r^5 \wedge C_{r'}^5; \mathbb{Z}/2)$ that $Sq^2 : H^7(Y; \mathbb{Z}/2) \rightarrow H^9(Y; \mathbb{Z}/2)$ is isomorphic. Hence Y is not a Moore space with nontrivial homology group at 7 or 8 dimension. So $Y \simeq *$ and $C_r^5 \wedge C_{r'}^5$ is indecomposable. \square

In the rest of this subsection we will give cell structure of spaces $M_{2^r}^n \wedge C_{\eta}^{n+2}$ ($n \geq 3$) and $C_{\eta}^5 \wedge C_r^{5,s}$ which will be used later.

Lemma 4.2. $M_{2^r}^n \wedge C_{\eta}^{n+2} \simeq S^{2n} \cup_{h^r} \mathbf{C} C^{(2n+2),r} \simeq C_r^{2n+2} \cup_{h_r} \mathbf{C} S^{2n+2}$ ($n \geq 3$), where h^r is determined by $h^r i_S = (2^r, \eta)$, i.e.,

$$\begin{array}{ccccccc} C^{(2n+2),r} & \xrightarrow{h^r} & S^{2n} & \xrightarrow{i_{\overline{C}}} & M_{2^r}^n \wedge C_{\eta}^{n+2} & \xrightarrow{q_{\overline{C}}} & C^{(2n+3),r} \\ \uparrow i_S & \nearrow h^r i_S = (2^r, \eta) & & & & & \\ S^{2n} \vee S^{2n+1} & & & & & & \end{array}$$

h_r is determined by $q_S h_r = \begin{pmatrix} \eta \\ 2^r \end{pmatrix}$, i.e.,

$$\begin{array}{ccccccc} S^{2n+2} & \xrightarrow{h_r} & C_r^{2n+2} & \xrightarrow{i_{\underline{C}}} & M_{2^r}^n \wedge C_{\eta}^{n+2} & \xrightarrow{q_{\underline{C}}} & S^{2n+3} \\ & \searrow q_S h_r = \begin{pmatrix} \eta \\ 2^r \end{pmatrix} & \downarrow q_S & & & & \\ & & S^{2n+1} \vee S^{2n+2} & & & & \end{array}$$

The top rows are cofibre sequences in each commutative diagrams.

Moreover,

$$\pi_{2n+1}(M_{2^r}^n \wedge C_{\eta}^{n+2}) = 0; \quad (2)$$

$$\mathbb{Z}/2^r \cong \frac{\pi_{2n+2} C_r^{2n+2}}{\langle (j_1 \eta)_{\overline{S}}, 2^{r-1} (2j_2)_{\overline{S}} \rangle} \xrightarrow[(i_{\underline{C}})_*]{\cong} \pi_{2n+2}(M_{2^r}^n \wedge C_{\eta}^{n+2}); \quad (3)$$

$$\frac{\pi_{2n+3} S^{2n}}{\langle 2^r q_{2n}, \eta^{(3)} \rangle} \xrightarrow[(i_{\overline{C}})_*]{\cong} \pi_{2n+3}(M_{2^r}^n \wedge C_{\eta}^{n+2}); \quad (4)$$

Dually

$$[M_{2^r}^n \wedge C_\eta^{n+2}, S^{2n+2}] = 0; \quad (5)$$

$$\mathbb{Z}/2^r \cong \frac{[C^{(2n+3),r}, S^{2n+1}]}{\langle (\eta p_2)_S^-, 2^{r-1}(2p_1)_S^- \rangle} \xrightarrow[(q_{\underline{C}})^*]{\cong} [M_{2^r}^n \wedge C_\eta^{n+2}, S^{2n+1}]; \quad (6)$$

$$\frac{[S^{2n+3}, S^{2n}]}{\langle 2^r \varrho_{2n}, \eta^{(3)} \rangle} \xrightarrow[(q_{\overline{C}})^*]{\cong} [M_{2^r}^n \wedge C_\eta^{n+2}, S^{2n}]; \quad (7)$$

where $\eta^{(3)} = \eta\eta\eta$.

Proof. From Lemma 2.4,

$$M_{2^r}^n \wedge C_\eta^{n+2} = S^n \wedge S^n \cup \mathbf{C}(S^n \wedge S^n \vee S^n \wedge S^{n+1}) \cup \mathbf{CC}(S^n \wedge S^{n+1});$$

$$(M_{2^r}^n \wedge C_\eta^{n+2})/S^{2n} \simeq (S^{2n+1} \vee S^{2n+2}) \cup \begin{pmatrix} \eta \\ -2^r \end{pmatrix} \mathbf{C}S^{2n+2} \simeq C^{(2n+3),r};$$

$$(M_{2^r}^n \wedge C_\eta^{n+2})^{(2n+2)} \simeq S^{2n} \cup_{(2^r, \eta)} \mathbf{C}(S^{2n} \vee S^{2n+1}) \simeq C_r^{2n+2}.$$

So, there are cofibre sequences

$$\begin{array}{ccccc} C^{(2n+2),r} & \xrightarrow{h^r} & S^{2n} & \xrightarrow{i_{\overline{C}}} & M_{2^r}^n \wedge C_\eta^{n+2}; \\ \uparrow i_S & \nearrow h^r i_S = (a, x\eta) & & & \\ S^{2n} \vee S^{2n+1} & & & & \end{array}$$

$$\begin{array}{ccccc} S^{2n+2} & \xrightarrow{h_r} & C_r^{2n+2} & \xrightarrow{i_{\underline{C}}} & M_{2^r}^n \wedge C_\eta^{n+2}, \\ \searrow q_S h_r = \begin{pmatrix} y\eta \\ b \end{pmatrix} & & \downarrow q_S & & \\ & & S^{2n+1} \vee S^{2n+2} & & \end{array}$$

where $a, b \in \mathbb{Z}$ and $x, y \in \{0, 1\}$. Since the following two homomorphisms

$$[C^{(2n+2),r}, S^{2n}] \xrightarrow{i_S^*} [S^{2n} \vee S^{2n+1}, S^{2n}]$$

$$[S^{2n+2}, C_r^{2n+2}] \xrightarrow{(q_S)^*} [S^{2n+2}, S^{2n+1} \vee S^{2n+2}]$$

are injective, h^r and h_r are determined by $h^r i_S$ and $q_S h_r$ respectively.

By $H_{2n}(M_{2^r}^n \wedge C_\eta^{n+2}) = \mathbb{Z}/2^r$ and $\pi_{2n+1}(M_{2^r}^n \wedge C_\eta^{n+2}) \cong \pi_{2n+1}(C_r^{2n+2}) = 0$,

$$a = 2^r, x = 1;$$

By $H_{2n+2}(M_{2^r}^n \wedge C_\eta^{n+2}) = \mathbb{Z}/2^r$ and $[M_{2^r}^n \wedge C_\eta^{n+2}, S^{2n+2}] \cong [C^{(2n+3),r}, S^{2n+2}] = 0$

$$b = 2^r, y = 1.$$

Thus we prove the first part of this Lemma. Now $\pi_*(M_{2^r}^n \wedge C_\eta^{n+2})(* = 2n+1, 2n+2, 2n+3)$ and $[M_{2^r}^n \wedge C_\eta^{n+2}, S^m](m = 2n, 2n+1, 2n+2)$ are easily obtained. \square

Lemma 4.3. $C_\eta^5 \wedge C_r^{5,s}$ is homotopy equivalent to the mapping cone of map

$$S^9 \xrightarrow{\begin{pmatrix} i_{\overline{C}} \varrho_6 \\ h_s \end{pmatrix}} M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \quad (i_{\overline{C}} \text{ and } h_s \text{ are defined in Lemma 4.2) and } \pi_9(C_\eta^5 \wedge C_r^{5,s}) \cong \begin{cases} \mathbb{Z}/2^{s+1} \oplus \mathbb{Z}/2, & r > 1 \\ \mathbb{Z}/2^s \oplus \mathbb{Z}/2, & r = 1 \end{cases}.$$

Proof. Apply Lemma 2.4 to cofibre sequences

$$S^4 \xrightarrow{\eta} S_a^3 \rightarrow C_\eta^5; \quad S^3 \vee M_{2^s}^3 \xrightarrow{f=(2^r, \eta q)} S_b^4 \rightarrow C_r^{5,s}.$$

$$(C_\eta^5 \wedge C_r^{5,s})/S^6 \simeq (\Sigma(S^4 \wedge S_b^3) \vee \Sigma(S_a^3 \wedge (S^3 \vee M_{2^s}^3))) \cup_{\mathcal{A}} \mathbf{C}\Sigma S^4 \wedge (S^3 \vee M_{2^s}^3)$$

where $\mathcal{A} = \begin{pmatrix} \Sigma 1 \wedge f \\ -\Sigma \eta \wedge 1 \end{pmatrix}$. i.e.,

$$(C_\eta^5 \wedge C_r^{5,s})/S^6 \simeq (S^8 \vee S^7 \vee M_{2^s}^7) \cup_{\mathcal{A}} \mathbf{C}(S^8 \vee M_{2^s}^8)$$

$$\text{where } \mathcal{A} = \begin{array}{c} S^8 \\ S^7 \\ M_{2^s}^7 \end{array} \begin{array}{|c|c|} \hline S^8 & M_{2^s}^8 \\ \hline 2^r & \eta q \\ \hline \eta & 0 \\ \hline 0 & \eta \wedge 1 \\ \hline \end{array} \xrightarrow[qr_3 + r_1]{\cong} \begin{array}{c} S^8 \\ S^7 \\ M_{2^s}^7 \end{array} \begin{array}{|c|c|} \hline S^8 & M_{2^s}^8 \\ \hline 2^r & 0 \\ \hline \eta & 0 \\ \hline 0 & \eta \wedge 1 \\ \hline \end{array}. \text{ Hence}$$

$$(C_\eta^5 \wedge C_r^{5,s})/S^6 \simeq M_{2^s}^4 \wedge C_\eta^5 \vee C^{9,r} \quad (8)$$

Apply Lemma 2.4 to cofibre sequences

$$S_1^4 \xrightarrow{\eta} S^3 \rightarrow C_\eta^5; \quad S_2^4 \xrightarrow{g=\begin{pmatrix} i\eta \\ 2^s \end{pmatrix}} M_{2^r}^3 \vee S_a^4 \rightarrow C_r^{5,s}.$$

$$\begin{aligned} (C_\eta^5 \wedge C_r^{5,s})^{(9)} &\simeq S^3 \wedge (M_{2^r}^3 \vee S_a^4) \cup_{(\eta \wedge 1, 1 \wedge g)} \mathbf{C}(S_1^4 \wedge (M_{2^r}^3 \vee S_a^4) \vee S^3 \wedge S_2^4) \\ &\simeq (M_{2^r}^6 \vee S^7) \cup_{\mathcal{B}} \mathbf{C}(M_{2^r}^7 \vee S^8 \vee S^7) \end{aligned}$$

$$\text{where } \mathcal{B} = \begin{array}{c} M_{2^r}^6 \\ S^7 \end{array} \begin{array}{|c|c|c|} \hline M_{2^r}^7 & S^8 & S^7 \\ \hline \eta \wedge 1 & 0 & i\eta \\ \hline 0 & \eta & 2^s \\ \hline \end{array} \xrightarrow[\mathbf{c}_1 i + \mathbf{c}_3]{\cong} \begin{array}{c} M_{2^r}^6 \\ S^7 \end{array} \begin{array}{|c|c|c|} \hline M_{2^r}^7 & S^8 & S^7 \\ \hline \eta \wedge 1 & 0 & 0 \\ \hline 0 & \eta & 2^s \\ \hline \end{array}. \text{ Thus}$$

$$(C_\eta^5 \wedge C_r^{5,s})^{(9)} \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \quad (9)$$

There is a cofibre sequence

$$S^9 \xrightarrow{\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}} M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \rightarrow C_\eta^5 \wedge C_r^{5,s} \rightarrow S^{10} \xrightarrow{\begin{pmatrix} \Sigma \alpha' \\ \Sigma \beta' \end{pmatrix}} M_{2^r}^4 \wedge C_\eta^5 \vee C_s^{10} \quad (10)$$

where $\alpha' = i_{\overline{C}}(t' \varrho_6)$, $t' = 1$ for $r = 1$ and $t' \in \{1, 2\}$ for $r > 1$ since $\pi_9(M_{2^r}^3 \wedge C_\eta^5) = \begin{cases} \mathbb{Z}/2 \langle i_{\overline{C}} \varrho_6 \rangle, & r = 1 \\ \mathbb{Z}/4 \langle i_{\overline{C}} \varrho_6 \rangle, & r \geq 2 \end{cases}$ and $C_\eta^5 \wedge C_r^{5,s}$ is indecomposable. β' is determined by

$q_S \beta' = \begin{pmatrix} y'\eta \\ b' \end{pmatrix}$ for some $y' \in \{0, 1\}$ and $b' \in \mathbb{Z}$ since $(q_S)_* : \pi_9 C_s^9 \rightarrow \pi_9(S^8 \vee S^9)$ is injective.

$$\begin{array}{ccc} S^6 & \xrightarrow{i_{\overline{C}}} & M_{2^r}^3 \wedge C_\eta^5 \\ \uparrow t' \varrho_6 & \nearrow \alpha' & \\ S^9 & & \end{array} \quad \begin{array}{ccc} S^9 & \xrightarrow{\beta'} & C_s^9 \\ & \searrow & \downarrow q_S \\ \begin{pmatrix} y'\eta \\ b' \end{pmatrix} & & S^8 \vee S^9 \end{array}$$

By $H_9(C_\eta^5 \wedge C_r^{5,s}) = \mathbb{Z}/2^s$, $b' = 2^s$. From $[C_\eta^5 \wedge C_r^{5,s}, S^9] \cong [(C_\eta^5 \wedge C_r^{5,s})/S^6, S^9] = [M_{2^s}^4 \wedge C_\eta^5 \vee C^{9,r}, S^9] \cong \mathbb{Z}/2^r$ and exact sequence $0 \rightarrow \frac{[S^{10}, S^9]}{\langle y'\eta \rangle} \rightarrow [C_\eta^5 \wedge C_r^{5,s}, S^9] \rightarrow \mathbb{Z}/2^r \rightarrow 0$ we have $y' = 1$. Thus $\beta' = h_s$. So for $r = 1$,

$$\pi_9(C_\eta^5 \wedge C_r^{5,s}) \cong \frac{\mathbb{Z}/4\langle i_{\overline{C}} \varrho_6 \rangle \oplus \mathbb{Z}/2\langle (i_1 \eta)^- \rangle \oplus \mathbb{Z}\langle (2i_2)^- \rangle}{\langle (t' i_{\overline{C}} \varrho_6, (i_1 \eta)^-, 2^{s-1}(2i_2)^-) \rangle} \cong \frac{\mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (1, 1, 2^{s-1}) \rangle} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2^s.$$

Next we will determine t' for $r > 1$.

By computing the exact sequence π_9 of cofibre sequence (10), we get

$$\begin{aligned} \pi_9(C_\eta^5 \wedge C_r^{5,s}) &\cong \frac{\mathbb{Z}/4\langle i_{\overline{C}} \varrho_6 \rangle \oplus \mathbb{Z}/2\langle (i_1 \eta)^- \rangle \oplus \mathbb{Z}\langle (2i_2)^- \rangle}{\langle (t' i_{\overline{C}} \varrho_6, (i_1 \eta)^-, 2^{s-1}(2i_2)^-) \rangle} \\ &\cong \frac{\mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (t', 1, 2^{s-1}) \rangle} \cong \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z}/2^s, & t' = 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s+1}, & t' = 1 \end{cases}. \end{aligned} \quad (11)$$

On the other hand, $\pi_9(C_\eta^5 \wedge C_r^{5,s}) \cong [C_\eta^{14}, C_r^{13,s}] \cong [C_\eta^7, C_r^{6,s}]$ ($[C_\eta^{k+1}, C_r^{k,s}]$ is stable for $k \geq 6$). There is an exact sequence

$$[C_\eta^7, S^5] \xrightarrow{\begin{pmatrix} i\eta \\ 2^s \end{pmatrix}} [C_\eta^7, M_{2^r}^4 \vee S^5] \rightarrow [C_\eta^7, C_r^{6,s}] \rightarrow [C_\eta^7, S^6] = 0.$$

From Lemma 3.1 and the following commutative diagram

$$\begin{array}{ccccccccc} [S^6, S^4] & \xrightarrow{\eta^*} & [S^7, S^4] & \xrightarrow{q_\eta^*} & [C_\eta^7, S^4] & \xrightarrow{0} & [S^5, S^4] & \xrightarrow[\cong]{\eta^*} & [S^6, S^4] \\ \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* \\ [S^6, M_{2^r}^4] & \xrightarrow{\eta^*} & [S^7, M_{2^r}^4] & \xrightarrow{q_\eta^*} & [C_\eta^7, M_{2^r}^4] & \xrightarrow{0} & [S^5, M_{2^r}^4] & \xrightarrow[\cong]{\eta^*} & [S^6, M_{2^r}^4] \end{array}$$

we get the surjection

$$\mathbb{Z}/12 \cong [C_\eta^7, S^4] \xrightarrow{i_*} [C_\eta^7, M_{2^r}^4] \cong \mathbb{Z}/4.$$

By $[C_\eta^7, C_\eta^6] \cong \frac{[C_\eta^7, S^4]}{\eta_*[C_\eta^7, S^5]} \cong \frac{[S^7, C_\eta^6]}{\eta_*[S^6, C_\eta^6]} \cong \mathbb{Z}/6$ ([16], Proposition 2.6 (iii)), we get

$$\mathbb{Z} = [C_\eta^7, S^5] \xrightarrow{\eta_* = 6} [C_\eta^7, S^4] = \mathbb{Z}/12.$$

So

$$(i\eta)_* = 2 : \mathbb{Z} = [C_\eta^7, S^5] \xrightarrow{\eta_*} [C_\eta^7, S^4] \xrightarrow{i_*} [C_\eta^7, M_{2r}^4] \cong \mathbb{Z}/4. \quad (12)$$

hence

$$\pi_9(C_\eta^5 \wedge C_r^{5,s}) \cong [C_\eta^7, C_r^{6,s}] \cong \frac{\mathbb{Z}/4 \oplus \mathbb{Z}}{\langle (2, 2^s) \rangle} \cong \mathbb{Z}/2^{s+1} \oplus \mathbb{Z}/2. \quad (13)$$

Together with (11), $t' = 1$. □

4.2 $M_{2u}^3 \wedge C$, $C \in \{C_r^5, C_r^{5,s}, C_r^{5,s} \mid r, s \in \mathbb{Z}^+\}$

(1) $M_{2u}^3 \wedge C_r^5$ and $M_{2u}^3 \wedge C_r^{5,s}$

There is the following commutative diagram

$$\begin{array}{ccccc} M_{2u}^3 \wedge (S^3 \vee S^4) & \xrightarrow{1 \wedge (2^r, \eta)} & M_{2u}^3 \wedge S^3 & \longrightarrow & M_{2u}^3 \wedge C_r^5 \\ \parallel & \nearrow (1 \wedge 2^r, 1 \wedge \eta) & & & \\ M_{2u}^3 \wedge S^3 \vee M_{2u}^3 \wedge S^4 & & & & \end{array}$$

where the top row is a cofibre sequence. Hence

$$M_{2u}^3 \wedge C_r^5 \simeq M_{2u}^3 \wedge S^3 \cup_{(1 \wedge 2^r, 1 \wedge \eta)} \mathbf{C}(M_{2u}^3 \wedge S^3 \vee M_{2u}^3 \wedge S^4)$$

- If $r \geq u$ and $r > 1$, then $1 \wedge 2^r \simeq 0$, hence $M_{2u}^3 \wedge C_r^5 \simeq M_{2u}^7 \vee M_{2u}^3 \wedge C_\eta^5$.
- If $r = u = 1$, $1 \wedge \eta \in [M_{2u}^7, M_{2u}^6]_\eta^\eta$ implies $1 \wedge \eta = \lambda_{11} + \varepsilon_u i\eta\eta q$ for some $\varepsilon_u \in \{0, 1\}$ (for $u = 1$, $\varepsilon_1 = 0$, i.e., $\lambda_{11} = 1 \wedge \eta$), then

$$M_2^3 \wedge C_1^5 \simeq M_2^6 \cup_{\mathcal{M}=(i_6\eta q_6, \lambda_{11})} \mathbf{C}(M_2^6 \vee M_2^7).$$

From $M_2^6 \begin{bmatrix} M_2^6 & M_2^7 \\ i_6\eta q_6 & \lambda_{11} \end{bmatrix} M_2^7 \begin{bmatrix} M_2^6 & M_2^7 \\ i_7q_6 & id \end{bmatrix} = M_2^6 \begin{bmatrix} M_2^6 & M_2^7 \\ 0 & \lambda_{11} \end{bmatrix}$, we get

$\mathcal{M} \cong (0, \lambda_{11})$, thus $M_2^3 \wedge C_1^5 \simeq M_2^7 \vee M_2^3 \wedge C_\eta^5$.

- If $r < u$, apply Lemma 2.4 to cofibre sequences

$$S_1^3 \xrightarrow{2^u} S_a^3 \rightarrow M_{2u}^3; \quad S_2^3 \vee S^4 \xrightarrow{(2^r, \eta)} S_b^3 \rightarrow C_r^5,$$

we can easily get

$$(M_{2u}^3 \wedge C_r^5)/S^6 \simeq S^7 \vee C_r^{9,u} \quad (14)$$

Suppose that $M_{2u}^3 \wedge C_r^5 \simeq X \vee Y$ are decomposable and $H_6(X) \neq 0$.

Since Sq^2 on $H^6(M_{2u}^3 \wedge C_r^5; \mathbb{Z}/2)$ and $H^7(M_{2u}^3 \wedge C_r^5; \mathbb{Z}/2)$ are nontrivial and

$$H_*(M_{2u}^3 \wedge C_r^5) = \begin{cases} \mathbb{Z}/2^r, & * = 6 \\ \mathbb{Z}/2^r, & * = 7 \\ \mathbb{Z}/2^u, & * = 8 \\ 0, & \text{otherwise} \end{cases},$$

$M_{2u}^3 \wedge C_r^5$ has no direct summands M_{2r}^6 and M_{2u}^8 . Thus $M_{2u}^3 \wedge C_r^5 \simeq X \vee M_{2r}^7$. By Lemma 2.7 and (14), $(X/S^6) \vee M_{2r}^7 \simeq S^7 \vee C_r^{9,u}$ which contradicts to the uniqueness of the decomposability of \mathbf{A}_n^3 -complexes [5]. So $M_{2u}^3 \wedge C_r^5$ is indecomposable for $r < u$.

In summary, $M_{2u}^3 \wedge C_r^5$ is homotopy equivalent to $M_{2u}^3 \wedge C_\eta^5 \vee M_{2u}^7$ for $r \geq u$ and indecomposable for $u > r$.

By the properties of the duality functor D we have

$M_{2u}^3 \wedge C^{5,s} \simeq D(M_{2u}^4 \wedge C_s^5)$ is homotopy equivalent to $M_{2u}^3 \wedge C_\eta^5 \vee M_{2u}^7$ for $s \geq u$ and indecomposable for $u > s$.

(2) $M_{2u}^3 \wedge C_r^{5,s}$

If $u > r$ and $u > s$ there is a cofibre sequence

$$S^3 \wedge C_r^{5,s} \xrightarrow{2^u \wedge 1} S^3 \wedge C_r^{5,s} \rightarrow M_{2u}^3 \wedge C_r^{5,s}$$

$[C_r^{k,s}, C_r^{k,s}] \cong \mathbb{Z}/2^{\max(r,s)+1} \oplus \mathbb{Z}/2^{m_{sr}} \oplus \mathbb{Z}/2^{m_{sr}+1}$ for $k \geq 5$ which implies $2^u \wedge 1 = 0 \in [S^3 \wedge C_r^{5,s}, S^3 \wedge C_r^{5,s}]$. Thus $M_{2u}^3 \wedge C_r^{5,s} \simeq C_r^{8,s} \vee C_r^{9,s}$.

If $u \leq r$ and $r > 1$, by **Cof4** of $C_r^{5,s}$,

$$\begin{array}{ccccc} M_{2u}^3 \wedge (S^3 \vee M_{2s}^3) & \xrightarrow{1 \wedge (2^r, \eta q)} & M_{2u}^3 \wedge S^3 & \longrightarrow & M_{2u}^3 \wedge C_r^{5,s} \\ \parallel & \nearrow (1 \wedge 2^r, 1 \wedge \eta q) & & & \\ M_{2u}^3 \wedge S^3 \vee M_{2u}^3 \wedge M_{2s}^3 & & & & \end{array}$$

where the top row is a cofibre sequence. Since $2^r = 0 \in [M_{2u}^6, M_{2u}^6]$,

$$M_{2u}^3 \wedge C_r^{5,s} \simeq M_{2u}^7 \vee \mathbf{C}_{1 \wedge \eta q} \simeq M_{2u}^3 \wedge C^{5,s} \vee M_{2u}^7$$

$$M_{2u}^3 \wedge C_r^{5,s} \simeq \begin{cases} M_{2u}^3 \wedge C^{5,s} \vee M_{2u}^7, & s < u \leq r \\ M_{2u}^3 \wedge C_\eta^5 \vee M_{2u}^7 \vee M_{2u}^7, & u \leq r, s \text{ and } r > 1 \end{cases}.$$

If $u \leq s$ and $s > 1$, $M_{2u}^3 \wedge C_r^{5,s} \simeq D(M_{2u}^4 \wedge C_s^{5r})$, we get

$$M_{2u}^3 \wedge C_r^{5,s} \simeq \begin{cases} M_{2u}^3 \wedge C_r^5 \vee M_{2u}^7, & r < u \leq s \\ M_{2u}^3 \wedge C_\eta^5 \vee M_{2u}^7 \vee M_{2u}^7, & u \leq r, s \text{ and } s > 1 \end{cases}.$$

If $u = r = s = 1$, then from Corollary 3.7 of [17],

$$M_2^3 \wedge C_1^{5,1} \simeq \Sigma M_2^3 \wedge M_2^1 \wedge M_2^1 \simeq M_2^3 \wedge C_\eta^5 \vee M_2^7 \vee M_2^7.$$

In summary,

$$M_{2^u}^3 \wedge C_r^{5,s} \simeq \begin{cases} C_r^{8,s} \vee C_r^{9,s}, & u > s \text{ and } u > r \\ M_{2^u}^3 \wedge C_r^5 \vee M_{2^u}^7, & r < u \leq s \\ M_{2^u}^3 \wedge C_r^{5,s} \vee M_{2^u}^7, & s < u \leq r \\ M_{2^u}^3 \wedge C_\eta^5 \vee M_{2^u}^7 \vee M_{2^u}^7, & u \leq r, s \end{cases}.$$

4.3 $C_u^5 \wedge C_r^{5,s}$ and $C^{5,u} \wedge C_r^{5,s}$ for $u, r, s \in \mathbb{Z}^+$

Let u_k be generators of $H^k(C_u^5; \mathbb{Z}/2)$ for $k = 3, 4, 5$ and $v_3, v_4, \overline{v_4}, v_5$ be generators of $H^*(C_r^{5,s}; \mathbb{Z}/2)$ which satisfy conditions (1) of Lemma 3.3.

$$H_*(C_u^5 \wedge C_r^{5,s}) = \begin{cases} \mathbb{Z}/2^{m_{ur}} & * = 6 \\ \mathbb{Z}/2^{m_{ur}} \oplus \mathbb{Z}/2^{m_{us}} & * = 7 \\ \mathbb{Z}/2^{m_{us}} \oplus \mathbb{Z}/2^r & * = 8 \\ \mathbb{Z}/2^s & * = 9 \\ 0 & \text{otherwise} \end{cases}$$

$$H^*(C_u^5 \wedge C_r^{5,s}; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\{u_3 \otimes v_3\} & * = 6 \\ \mathbb{Z}/2\{u_3 \otimes v_4, u_3 \otimes \overline{v_4}, u_4 \otimes v_3\} & * = 7 \\ \mathbb{Z}/2\{u_3 \otimes v_5, u_4 \otimes v_4, u_4 \otimes \overline{v_4}, u_5 \otimes v_3\} & * = 8 \\ \mathbb{Z}/2\{u_4 \otimes v_5, u_5 \otimes v_4, u_5 \otimes \overline{v_4}\} & * = 9 \\ \mathbb{Z}/2\{u_5 \otimes v_5\} & * = 10 \\ 0 & \text{otherwise} \end{cases}$$

The Steenrod operation action on $H^*(C_u^5 \wedge C_r^{5,s}; \mathbb{Z}/2)$ is given as follows

- (i) $Sq^4(u_3 \otimes v_3) = u_5 \otimes v_5;$
- (ii) $Sq^2(u_3 \otimes v_5) = Sq^2(u_5 \otimes v_3) = u_5 \otimes v_5;$
- (iii) $Sq^2(u_3 \otimes v_3) = \begin{cases} u_3 \otimes v_5 + u_4 \otimes v_4 + u_5 \otimes v_3, & u = r = 1 \\ u_3 \otimes v_5 + u_5 \otimes v_3, & u > 1 \text{ or } r > 1 \end{cases};$
- (iv) $Sq^2(u_3 \otimes v_4) = u_5 \otimes v_4;$
 $Sq^2(u_4 \otimes v_3) = u_4 \otimes v_5;$
 $Sq^2(u_3 \otimes \overline{v_4}) = \begin{cases} u_5 \otimes \overline{v_4} + u_4 \otimes v_5, & u = s = 1 \\ u_5 \otimes \overline{v_4}, & u > 1 \text{ or } s > 1 \end{cases}.$

Corollary 4.4. *If $C_u^5 \wedge C_r^{5,s} \simeq X \vee Y$ is decomposable and $H_6(X) \neq 0$, then X is indecomposable and $Y \simeq C_l^{9,t}$ for some $t \in \{m_{us}, r\}, l \in \{m_{ur}, m_{us}\}$.*

Proof. From Lemma 2.6, $H_t X \cong H_t(C_u^5 \wedge C_r^{5,s})$ for $t = 6, 9$ and $\dim H_7 X + \dim H_8 X \geq 2$. Neither $M_{2^x}^7$ nor $M_{2^x}^8$ can be a summand of Y for any $x \in \mathbb{Z}^+$ since

$$Sq^2 : H^7(C_u^5 \wedge C_r^{5,s}; \mathbb{Z}/2) \rightarrow H^9(C_u^5 \wedge C_r^{5,s}; \mathbb{Z}/2)$$

is isomorphic. Thus $\dim H_7 X = \dim H_8 X = 1$ and $\dim H_* Y = \begin{cases} 1, * = 7, 8 \\ 0, \text{otherwise} \end{cases}$ which implies that X is indecomposable and $Y \simeq C_l^{9,t}$ for some $t \in \{m_{us}, r\}, l \in \{m_{ur}, m_{us}\}$. \square

Firstly we study $(C_u^5 \wedge C_r^{5,s})/S^6$.

Applying Lemma 2.4 to the following cofibre sequences

$$S_1^3 \vee S^4 \xrightarrow{(2^u, \eta)} S_a^3 \rightarrow C_u^5; \quad S_2^3 \vee M_{2^s}^3 \xrightarrow{(2^r, \eta q)} S_b^3 \rightarrow C_r^{5,s},$$

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq (\Sigma(S_1^3 \vee S^4) \wedge S_b^3 \vee \Sigma S_a^3 \wedge (S_2^3 \vee M_{2^s}^3)) \cup_{\mathcal{A}} \mathbf{C}\Sigma(S_1^3 \vee S^4) \wedge (S_2^3 \vee M_{2^s}^3),$$

where $\mathcal{A} = \begin{pmatrix} \Sigma 1 \wedge (2^r, \eta q) \\ -\Sigma(2^u, \eta) \wedge 1 \end{pmatrix}$, i.e.,

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq (S^7 \vee S^8 \vee S^7 \vee M_{2^s}^7) \cup_{\mathcal{A}} \mathbf{C}(S^7 \vee S^8 \vee M_{2^s}^7 \vee M_{2^s}^8)$$

$$\mathcal{A} = \begin{array}{c} S^7 \\ S^8 \\ S^7 \\ M_{2^s}^7 \end{array} \begin{array}{c} S^7 \quad S^8 \quad M_{2^s}^7 \quad M_{2^s}^8 \\ \begin{array}{|c|c|c|c|} \hline 2^r & 0 & \eta q & 0 \\ \hline 0 & 2^r & 0 & \eta q \\ \hline -2^u & \eta & 0 & 0 \\ \hline 0 & 0 & -2^u & \eta \wedge 1 \\ \hline \end{array} \end{array}.$$

Note that $q(\eta \wedge 1) = \eta q$ for $u \geq s$; $2^u = \begin{cases} 0, & u > 1 \\ i\eta q, & u = s = 1 \end{cases}$ in $[M_{2^s}^7, M_{2^s}^7]$.

- (i) For $s \leq u < r$, by transformations $i\mathbf{r}_1 + \mathbf{r}_4$ if $u = 1$ (otherwise, omitting this one); $q\mathbf{r}_4 + \mathbf{r}_2$; $2^{r-u}\mathbf{r}_3 + \mathbf{r}_1$ and $-\mathbf{r}_3$,

$$\mathcal{A} \cong \begin{array}{c} S^7 \\ S^8 \\ S^7 \\ M_{2^s}^7 \end{array} \begin{array}{c} S^7 \quad S^8 \quad M_{2^s}^7 \quad M_{2^s}^8 \\ \begin{array}{|c|c|c|c|} \hline 0 & 0 & \eta q & 0 \\ \hline 0 & 2^r & 0 & 0 \\ \hline 2^u & \eta & 0 & 0 \\ \hline 0 & 0 & 0 & \eta \wedge 1 \\ \hline \end{array} \end{array}$$

Thus

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C_u^{9,r} \vee C^{9,s} \vee M_{2^s}^4 \wedge C_\eta^5. \quad (15)$$

- (ii) For $s \leq u$ and $r \leq u$, by transformations $i\mathbf{r}_1 + \mathbf{r}_4$ if $u = 1$; $q\mathbf{r}_4 + \mathbf{r}_2$; $2^{u-r}\mathbf{r}_1 + \mathbf{r}_3$; $\mathbf{c}_2 q + \mathbf{c}_3$ if $r = u$,

$$\mathcal{A} \cong \begin{array}{c} S^7 \\ S^8 \\ S^7 \\ M_{2^s}^7 \end{array} \begin{array}{c} S^7 \quad S^8 \quad M_{2^s}^7 \quad M_{2^s}^8 \\ \begin{array}{|c|c|c|c|} \hline 2^r & 0 & \eta q & 0 \\ \hline 0 & 2^r & 0 & 0 \\ \hline 0 & \eta & 0 & 0 \\ \hline 0 & 0 & 0 & \eta \wedge 1 \\ \hline \end{array} \end{array}$$

Thus

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C_r^{9,s} \vee C^{9,r} \vee M_{2^s}^4 \wedge C_\eta^5. \quad (16)$$

(iii) For $r \leq u < s$ by transformations $q\mathbf{r}_4 + \mathbf{r}_2$; $2^{u-r}\mathbf{c}_2q + \mathbf{c}_3$; $2^{u-r}\mathbf{r}_1 + \mathbf{r}_3$ and $-\mathbf{r}_4$,

$$\mathcal{A} \cong \begin{array}{c} S^7 \\ S^8 \\ S^7 \\ M_{2^s}^7 \end{array} \begin{array}{c|ccc} S^7 & S^8 & M_{2^s}^7 & M_{2^s}^8 \\ \hline 2^r & 0 & \eta q & 0 \\ 0 & 2^r & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & 2^u & \eta \wedge 1 \end{array}.$$

Thus

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C^{9,r} \vee Z \quad (17)$$

$$\text{where } Z = (S^7 \vee M_{2^s}^7) \cup \begin{pmatrix} 2^r & \eta q & 0 \\ 0 & 2^u & \eta \wedge 1 \end{pmatrix} \mathbf{C}(S^7 \vee M_{2^s}^7 \vee M_{2^s}^8).$$

(iv) For $u < s$ and $u < r$, by transformations $2^{r-u}\mathbf{r}_3 + \mathbf{r}_1$; $-\mathbf{r}_3$ and $-\mathbf{r}_4$,

$$\mathcal{A} \cong \begin{array}{c} S^7 \\ S^8 \\ S^7 \\ M_{2^s}^7 \end{array} \begin{array}{c|ccc} S^7 & S^8 & M_{2^s}^7 & M_{2^s}^8 \\ \hline 0 & 0 & \eta q & 0 \\ 0 & 2^r & 0 & \eta q \\ 2^u & \eta & 0 & 0 \\ 0 & 0 & 2^u & \eta \wedge 1 \end{array}. \quad (18)$$

Secondly, we study the codimension 1 skeleton $(C_u^5 \wedge C_r^{5,s})^{(9)}$ of $C_u^5 \wedge C_r^{5,s}$.
Using cofibre sequences

$$S_1^4 \xrightarrow{i\eta} M_{2^u}^3 \rightarrow C_u^5; \quad S_2^4 \xrightarrow{g=\begin{pmatrix} i\eta \\ 2^s \end{pmatrix}} M_{2^r}^3 \vee S_b^4 \rightarrow C_r^{5,s}$$

$$(C_u^5 \wedge C_r^{5,s})^{(9)} \simeq M_{2^u}^3 \wedge (M_{2^r}^3 \vee S_b^4) \cup_{\mathcal{B}=(i\eta \wedge 1, 1 \wedge g)} \mathbf{C}(S_1^4 \wedge (M_{2^r}^3 \vee S_b^4) \vee M_{2^u}^3 \wedge S_2^4)$$

i.e., $(C_u^5 \wedge C_r^{5,s})^{(9)} \simeq (M_{2^u}^3 \wedge S_b^4 \vee M_{2^u}^3 \wedge M_{2^r}^3) \cup_{\mathcal{B}} \mathbf{C}(S_1^4 \wedge S_b^4 \vee S_1^4 \wedge M_{2^r}^3 \vee M_{2^u}^3 \wedge S_2^4)$,
where

$$\mathcal{B} = \begin{array}{c} M_{2^u}^3 \wedge S_b^4 \\ M_{2^u}^3 \wedge M_{2^r}^3 \end{array} \begin{array}{c|ccc} S_1^4 \wedge S_b^4 & S_1^4 \wedge M_{2^r}^3 & M_{2^u}^3 \wedge S_2^4 \\ \hline i\eta \wedge 1 & i\eta \wedge 0 & 1 \wedge 2^s \\ i\eta \wedge 0 & i\eta \wedge 1 & 1 \wedge i\eta \end{array}.$$

Let

$$\mathcal{B}_1 = \begin{array}{c} M_{2^u}^3 \wedge M_{2^r}^3 \end{array} \begin{array}{c|ccc} S_1^4 \wedge M_{2^r}^3 & M_{2^u}^3 \wedge S_2^4 \\ \hline i\eta \wedge 1 & 1 \wedge i\eta \end{array}.$$

For $u \geq r$ and $u > 1$, by Corollary 3.2, there is a retraction τ'_3 of $i \wedge 1$ yielding the following commutative diagram

$$\begin{array}{ccccc}
& & S^4 \wedge M_{2r}^3 & & \\
& \swarrow \eta \wedge 1 & \downarrow i\eta \wedge 1 & \searrow 0 & \\
S^3 \wedge M_{2r}^3 & \xrightarrow{i \wedge 1} & M_{2u}^3 \wedge M_{2r}^3 & \xrightarrow{q \wedge 1} & S^4 \wedge M_{2r}^3 \\
& \xleftarrow{\tau'_3} & & &
\end{array}$$

From the following commutative diagrams

$$\begin{array}{ccc}
M_{2u}^3 \wedge S^4 & \xrightarrow{q \wedge 1} & S^4 \wedge S^4 \\
\downarrow 1 \wedge \eta & & \downarrow 1 \wedge \eta \\
M_{2u}^3 \wedge S^3 & \xrightarrow{q \wedge 1} & S^4 \wedge S^3 \\
\downarrow 1 \wedge i & & \downarrow 1 \wedge i \\
M_{2u}^3 \wedge M_{2r}^3 & \xrightarrow{q \wedge 1} & S^4 \wedge M_{2r}^3
\end{array}
\quad
\begin{array}{ccccc}
& S^3 \wedge S^3 & \xrightarrow{i \wedge 1} & M_{2u}^3 \wedge S^3 & \xrightarrow{q \wedge 1} & S^4 \wedge S^3 \\
& \downarrow 1 \wedge i & & \downarrow 1 \wedge i & & \\
1 \curvearrowright & S^3 \wedge M_{2r}^3 & \xrightarrow{i \wedge 1} & M_{2u}^3 \wedge M_{2r}^3 & & \\
& \searrow & \searrow & \downarrow \tau'_3 & & \\
& S^3 \wedge S^3 & \xrightarrow{1 \wedge i} & S^3 \wedge M_{2r}^3 & &
\end{array}$$

we get

$$(q \wedge 1)(1 \wedge i\eta) = i\eta q \in [M_{2u}^7, M_{2r}^7]; \quad \tau'_3(1 \wedge i) \in [M_{2u}^6, M_{2r}^6]_1^{2^{u-r}}.$$

which implies $\tau'_3(1 \wedge i\eta) \in [M_{2u}^7, M_{2r}^6]_\eta$ for $u = r$ and $\tau'_3(1 \wedge i\eta) \in [M_{2u}^7, M_{2r}^6]_\eta^0$ for $u > r$. Since $\eta \wedge 1 : S^4 \wedge M_{2r}^3 \rightarrow S^3 \wedge M_{2r}^3$ is also an element in $[M_{2r}^7, M_{2r}^6]_\eta$,

$$\tau'_3(1 \wedge i\eta) = \begin{cases} \eta \wedge 1 + \kappa i\eta\eta q, & r = u > 1 \\ \eta_r^u + \kappa i\eta\eta q, & u > r \end{cases}.$$

For $r > u \geq 1$, similarly, there is a retraction τ_3 of $1 \wedge i$ yielding the following commutative diagram

$$\begin{array}{ccccc}
& & M_{2u}^3 \wedge S^4 & & \\
& \swarrow 1 \wedge \eta & \downarrow 1 \wedge i\eta & \searrow 0 & \\
M_{2u}^3 \wedge S^3 & \xrightarrow{1 \wedge i} & M_{2u}^3 \wedge M_{2r}^3 & \xrightarrow{1 \wedge q} & M_{2u}^3 \wedge S^4 \\
& \xleftarrow{\tau_3} & & &
\end{array}$$

and for $i\eta \wedge 1 : S^4 \wedge M_{2r}^3 \rightarrow M_{2u}^3 \wedge M_{2r}^3$ we have

$$\tau_3(i\eta \wedge 1) = \eta_u^r + \kappa i\eta\eta q \in [M_{2r}^7, M_{2u}^6]; \quad (1 \wedge q)(i\eta \wedge 1) = i\eta q \in [M_{2r}^7, M_{2u}^7].$$

Note that for $r > u$, the composition of $M_{2r}^7 \xrightarrow{B(\chi)} M_{2u}^7 \xrightarrow{1 \wedge \eta} M_{2u}^6$ is an element in $[M_{2r}^7, M_{2u}^6]_\eta^0$, hence

$$(1 \wedge \eta)B(\chi) = \eta_u^r + \kappa' i\eta\eta q$$

and similarly, for the composition of $M_{2u}^7 \xrightarrow{B(\chi)} M_{2r}^7 \xrightarrow{\eta \wedge 1} M_{2r}^6$ ($u > r$), we have

$$(\eta \wedge 1)B(\chi) = \eta_r^u + \kappa' i\eta\eta q.$$

From the calculations above

$$\mathcal{B}_1 = \begin{cases} \begin{array}{c} M_{2u}^6 \quad \begin{array}{cc} M_{2r}^7 & M_{2u}^7 \\ \eta_r^u + \kappa i\eta\eta q & 1 \wedge \eta \\ i\eta q & 0 \end{array} \\ M_{2u}^7 \end{array} \xrightarrow[\text{Tr1}]{\cong} \begin{array}{c} M_{2u}^6 \quad \begin{array}{cc} M_{2r}^7 & M_{2u}^7 \\ 0 & 1 \wedge \eta \\ i\eta q & 0 \end{array} \\ M_{2u}^7 \end{array}, & r > u \geq 1 \\ \begin{array}{c} M_{2r}^6 \quad \begin{array}{cc} M_{2r}^7 & M_{2u}^7 \\ \eta \wedge 1 & \eta \wedge 1 + \kappa i\eta\eta q \\ 0 & i\eta q \end{array} \\ M_{2r}^7 \end{array} \xrightarrow[\text{Tr2}]{\cong} \begin{array}{c} M_{2r}^6 \quad \begin{array}{cc} M_{2r}^7 & M_{2u}^7 \\ \eta \wedge 1 & 0 \\ 0 & i\eta q \end{array} \\ M_{2r}^7 \end{array}, & r = u > 1 \\ \begin{array}{c} M_{2r}^6 \quad \begin{array}{cc} M_{2r}^7 & M_{2u}^7 \\ \eta \wedge 1 & \eta_r^u + \kappa i\eta\eta q \\ 0 & i\eta q \end{array} \\ M_{2r}^7 \end{array} \xrightarrow[\text{Tr3}]{\cong} \begin{array}{c} M_{2r}^6 \quad \begin{array}{cc} M_{2r}^7 & M_{2u}^7 \\ \eta \wedge 1 & 0 \\ 0 & i\eta q \end{array} \\ M_{2r}^7 \end{array}, & 1 \leq r < u \\ C_1^{8,1} \quad \begin{array}{cc} M_2^7 & M_2^7 \\ 1 \wedge i\eta & i\eta \wedge 1 \end{array}, & r = u = 1. \end{cases} \quad (19)$$

where the invertible transformations are given by

$$\text{Tr1} : \mathbf{c}_2(-B(\chi) - (\kappa + \kappa')i\eta q) + \mathbf{c}_1;$$

$$\text{Tr2} : \mathbf{c}_1(1 + \kappa i\eta q) + \mathbf{c}_2;$$

$$\text{Tr3} : \mathbf{c}_1(-B(\chi) - (\kappa + \kappa')i\eta q) + \mathbf{c}_2.$$

- For $s \geq u$, since $2^s = 0 \in [M_{2u}^7, M_{2u}^7]$ for $s > 1$ and take invertible transformation $\mathbf{c}_1 q + \mathbf{c}_3$ on \mathcal{B} for $s = 1$, we get

$$\mathcal{B} \cong M_{2u}^7 \quad \begin{array}{c} S^8 \\ i\eta \end{array} \oplus \mathcal{B}_1, \quad (20)$$

So

$$(C_u^5 \wedge C_r^{5,s})^{(9)} \simeq \mathbf{C}_{\mathcal{B}} \simeq \mathbf{C}_{i\eta} \vee \mathbf{C}_{\mathcal{B}_1} = C_u^9 \vee \mathbf{C}_{\mathcal{B}_1} \quad (21)$$

Lemma 4.5. *The mapping cone of the map $C_1^{8,1} \quad \begin{array}{cc} M_2^7 & M_2^7 \\ 1 \wedge i\eta & i\eta \wedge 1 \end{array}$ is homotopy equivalent to $M_2^3 \wedge C_\eta^5 \vee C_1^{9,1}$.*

Proof. $\Sigma C_1^5 \wedge C_1^{5,1} \simeq C_1^5 \wedge M_2^3 \wedge M_2^1 \simeq (M_2^3 \wedge C_\eta^5 \vee M_2^7) \wedge M_2^1 \simeq C_1^{6,1} \wedge C_\eta^5 \vee C_1^{10,1}$, hence $C_1^5 \wedge C_1^{5,1} \simeq C_1^{5,1} \wedge C_\eta^5 \vee C_1^{9,1}$. Together with $(C_1^{5,1} \wedge C_\eta^5)^{(9)} \simeq M_2^3 \wedge C_\eta^5 \vee C_1^9$ (from Lemma 4.3), we have

$$(C_1^5 \wedge C_1^{5,1})^{(9)} \simeq M_2^3 \wedge C_\eta^5 \vee C_1^9 \vee C_1^{9,1}.$$

On the other hand, from (21),

$$(C_1^5 \wedge C_1^{5,1})^{(9)} \simeq C_1^9 \vee C_1^{8,1} \cup_{(1 \wedge i\eta, i\eta \wedge 1)} \mathbf{C}(M_2^7 \vee M_2^7)$$

So by Lemma 2.7, $C_1^{8,1} \cup_{(1 \wedge i\eta, i\eta \wedge 1)} \mathbf{C}(M_2^7 \vee M_2^7) \simeq M_2^3 \wedge C_\eta^5 \vee C_1^{9,1}$. \square

Now, from (19), (21) and Lemma 4.5 we get

$$\text{if } s \geq u \text{ and } r \geq u, \quad \text{then } (C_u^5 \wedge C_r^{5,s})^{(9)} \simeq C_u^9 \vee C_u^{9,r} \vee M_{2^u}^3 \wedge C_\eta^5; \quad (22)$$

$$\text{if } s \geq u > r, \quad \text{then } (C_u^5 \wedge C_r^{5,s})^{(9)} \simeq C_u^9 \vee C_r^{9,u} \vee M_{2^r}^3 \wedge C_\eta^5. \quad (23)$$

- For $u > s$, it is easy to get

if $u > s$ and $u \geq r$, then

$$\mathcal{B} \cong \begin{array}{c} M_{2^u}^7 \\ M_{2^r}^6 \\ M_{2^r}^7 \end{array} \begin{array}{|ccc|} \hline S^8 & M_{2^r}^7 & M_{2^u}^7 \\ \hline i\eta & 0 & 2^s \\ 0 & \eta \wedge 1 & 0 \\ 0 & 0 & i\eta q \\ \hline \end{array},$$

hence

$$(C_u^5 \wedge C_r^{5,s})^{(9)} \simeq M_{2^r}^3 \wedge C_\eta^5 \vee Z', \quad (24)$$

where $Z' = (M_{2^u}^7 \vee M_{2^r}^7) \cup \begin{pmatrix} i\eta, 2^s \\ 0, i\eta q \end{pmatrix} \mathbf{C}(S^8 \vee M_{2^u}^7)$;

if $r > u > s$, then

$$\mathcal{B} = \begin{array}{c} M_{2^u}^7 \\ M_{2^u}^6 \\ M_{2^u}^7 \end{array} \begin{array}{|ccc|} \hline S^8 & M_{2^r}^7 & M_{2^u}^7 \\ \hline i\eta & 0 & 2^s \\ 0 & \eta_u^r + \kappa i\eta\eta q & 1 \wedge \eta \\ 0 & i\eta q & 0 \\ \hline \end{array}. \quad (25)$$

The decomposability of $C_u^5 \wedge C_r^{5,s}$ can be obtained from the structure of $(C_u^5 \wedge C_r^{5,s})/S^6$ and $(C_u^5 \wedge C_r^{5,s})^{(9)}$ now.

- (i) For $s < u < r$, suppose $C_u^5 \wedge C_r^{5,s}$ is decomposable, by Corollary 4.4 and (15), $C_u^5 \wedge C_r^{5,s} \simeq X \vee C_u^{9,r}$. However, $\pi_8 C_u^{9,r} \cong \mathbb{Z}/2^{r+1}$, which is not a direct summand of $\pi_8(C_u^5 \wedge C_r^{5,s}) \cong [C_u^{5,u}, C_r^{5,s}] \cong \mathbb{Z}/2^{s+1} \oplus \mathbb{Z}/2^r$, hence $C_u^5 \wedge C_r^{5,s}$ is indecomposable for $s < u < r$.
- (ii) For $u < s, u < r$, suppose $C_u^5 \wedge C_r^{5,s}$ is decomposable, by Corollary 4.4 and (22), $C_u^5 \wedge C_r^{5,s} \simeq X \vee C_u^{9,r}$. There is a cofibre sequence,

$$S^7 \vee S^8 \vee M_{2^s}^7 \vee M_{2^s}^8 \xrightarrow{\mathcal{A}} S^7 \vee S^8 \vee S^7 \vee M_{2^s}^7 \rightarrow (C_u^5 \wedge C_r^{5,s})/S^6 \rightarrow S^8 \vee S^9 \vee M_{2^s}^8 \vee M_{2^s}^9 \xrightarrow{\Sigma\mathcal{A}} S^8 \vee S^9 \vee S^8 \vee M_{2^s}^8, \text{ where } \mathcal{A} \text{ is the map (18). So,}$$

$$\mathbb{Z} \oplus \mathbb{Z}/2^s \xrightarrow{[\Sigma\mathcal{A}, S^9]} \mathbb{Z} \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2 \rightarrow [(C_u^5 \wedge C_r^{5,s})/S^6, S^9] \rightarrow 0$$

where $[\Sigma\mathcal{A}, S^9] = \begin{matrix} \mathbb{Z} & \mathbb{Z}/2^s \\ \mathbb{Z}/2^s & \begin{matrix} 2^r & 0 \\ 0 & 2^u \\ \mathbb{Z}/2 & 1 & 1 \end{matrix} \end{matrix}$ ($k : A_1 \rightarrow A_2$ denotes the homomorphism of abelian groups defined by multiplication by k). Thus

$$\begin{aligned} [C_u^5 \wedge C_r^{5,s}, S^9] &\cong [(C_u^5 \wedge C_r^{5,s})/S^6, S^9] \cong \frac{\mathbb{Z} \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2}{\langle (2^r, 0, 1), (0, 2^u, 1) \rangle} \\ &\cong \frac{\mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^s}{\langle (2^r, 2^u) \rangle} \cong \frac{\mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^s}{\langle (0, 2^u) \rangle} \cong \mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^u, \end{aligned}$$

which is a contradiction since $[C_u^{9,r}, S^9] \cong \mathbb{Z}/2^r$. Hence $C_u^5 \wedge C_r^{5,s}$ is indecomposable for $u < s$ and $u < r$.

(iii) For $r \leq u < s$

Lemma 4.6. *The wedge summand*

$$Z = (S^7 \vee M_{2^s}^7) \cup \begin{pmatrix} 2^r & \eta q & 0 \\ 0 & 2^u & \eta \wedge 1 \end{pmatrix} \mathbf{C}(S^7 \vee M_{2^s}^7 \vee M_{2^s}^8)$$

in (17) is indecomposable.

Proof. Assume that $Z \simeq Z_1 \vee Z_2$ is decomposable and $H_9 Z_1 \neq 0$. From the mapping cone structure of Z , we get $[Z, S^9] \cong \frac{\mathbb{Z}/2^s \oplus \mathbb{Z}/2}{\langle (2^u, 1) \rangle} \cong \mathbb{Z}/2^{u+1}$. For the pinch map $P : C_u^5 \wedge C_r^{5,s} \rightarrow (C_u^5 \wedge C_r^{5,s})/S^6$, there are isomorphisms

$$P^* : H^*((C_u^5 \wedge C_r^{5,s})/S^6; \mathbb{Z}/2) \xrightarrow{\cong} H^*(C_u^5 \wedge C_r^{5,s}; \mathbb{Z}/2) \quad (* = 7, 8, 9)$$

Thus $Sq^2 : H^7((C_u^5 \wedge C_r^{5,s})/S^6; \mathbb{Z}/2) \rightarrow H^9((C_u^5 \wedge C_r^{5,s})/S^6; \mathbb{Z}/2)$ is isomorphic and Sq^2 on $H^7(Z; \mathbb{Z}/2)$ is also isomorphic which implies that Moore spaces can not be split out of Z . Together with

$$H_* Z = \begin{cases} \mathbb{Z}/2^r \oplus \mathbb{Z}/2^u, & * = 7 \\ \mathbb{Z}/2^u, & * = 8 \\ \mathbb{Z}/2^s, & * = 9 \\ 0, & \text{otherwise} \end{cases},$$

we get $Z_2 = C_u^{9,u}$ or $C_r^{9,u}$. Thus $[Z_2, S^9] \cong \mathbb{Z}/2^u$ which contradicts to $[Z, S^9] \cong \mathbb{Z}/2^{u+1}$. \square

Now assume $C_u^5 \wedge C_r^{5,s}$ is decomposable, by Corollary 4.4 and its homology,

$$C_u^5 \wedge C_r^{5,s} \simeq X \vee C_l^{9,k}, \quad k, l \in \{u, r\}.$$

Hence $C_l^{9,k}$ is split out of $(C_u^5 \wedge C_r^{5,s})/S^6$, which is a contradiction since $(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C^{9,r} \vee Z$ and Z is indecomposable. Thus $C_u^5 \wedge C_r^{5,s}$ is indecomposable for $r \leq u < s$.

(iv) For $u \geq s, u \geq r$

Lemma 4.7. *The wedge summand Z' in (24) is homotopy equivalent to $C_r^{9,s} \vee C_s^9$, i.e.,*

$$(M_{2^u}^7 \vee M_{2^r}^7) \cup \begin{pmatrix} i\eta, 2^s \\ 0, i\eta q \end{pmatrix} \mathbf{C}(S^8 \vee M_{2^u}^7) \simeq C_r^{9,s} \vee C_s^9 \text{ for } u > s, u \geq r.$$

Proof. Let $W := (M_{2^u}^7 \vee M_{2^r}^7) \cup \begin{pmatrix} i\eta & 2^s i \\ 0 & 0 \end{pmatrix} \mathbf{C}(S^8 \vee S^7) = M_{2^r}^7 \vee U$, where

$$U := M_{2^u}^7 \cup_{(i\eta, 2^s i)} \mathbf{C}(S^8 \vee S^7)$$

Since $U^{(8)} = M_{2^u}^7 \cup_{2^s i} \mathbf{C}S^7 \simeq M_{2^s}^7 \vee S^8$, there is a cofibre sequence

$$S^8 \xrightarrow{\begin{pmatrix} xi\eta \\ a \end{pmatrix}} M_{2^s}^7 \vee S^8 \rightarrow U \text{ for some } x \in \{0, 1\} \text{ and } a \in \mathbb{Z}. \text{ By } H_8 U = \mathbb{Z} \text{ and } \pi_8 U = \mathbb{Z}, \text{ we have } x = 1 \text{ and } a = 0. \text{ Hence}$$

$$U \simeq S^8 \vee C_s^9, \quad W \simeq M_{2^r}^7 \vee S^8 \vee C_s^9. \quad (26)$$

There is a commutative diagram

$$\begin{array}{ccccccc} S^8 \vee S^7 & \xrightarrow{\begin{pmatrix} i\eta, 2^s i \\ 0, 0 \end{pmatrix}} & M_{2^u}^7 \vee M_{2^r}^7 & \longrightarrow & W & \longrightarrow & S^9 \vee S^8 \longrightarrow M_{2^u}^8 \vee M_{2^r}^8 \\ \begin{pmatrix} 1, 0 \\ 0, i \end{pmatrix} \downarrow & & \parallel & & \downarrow & & \downarrow & & \parallel \\ S^8 \vee M_{2^u}^7 & \xrightarrow{\begin{pmatrix} i\eta, 2^s \\ 0, i\eta q \end{pmatrix}} & M_{2^u}^7 \vee M_{2^r}^7 & \longrightarrow & Z' & \longrightarrow & S^9 \vee M_{2^u}^8 & \longrightarrow & M_{2^u}^8 \vee M_{2^r}^8 \end{array}$$

$$H_*(Z'/W) = \begin{cases} 0, & \text{otherwise} \\ \mathbb{Z}, & * = 9. \end{cases} \quad \text{i.e., } Z'/W \simeq S^9. \quad (27)$$

Thus there is a cofibre sequence

$$S^8 \xrightarrow{\begin{pmatrix} y i\eta \\ b \\ 0 \end{pmatrix}} M_{2^r}^7 \vee S^8 \vee C_s^9 \rightarrow Z' \rightarrow S^9.$$

for some $y \in \{0, 1\}$ and $b \in \mathbb{Z}$. From $H_8(Z') = \mathbb{Z}/2^s$, $b = 2^s$.

Inclusion $J : (C_u^5 \wedge C_r^{5,s})^{(9)} \rightarrow C_u^5 \wedge C_r^{5,s}$ induces isomorphisms

$$J^* : H^*(C_u^5 \wedge C_r^{5,s}; \mathbb{Z}/2) \xrightarrow{\cong} H^*((C_u^5 \wedge C_r^{5,s})^{(9)}; \mathbb{Z}/2) \quad * = 7, 8, 9.$$

Thus Sq^2 is isomorphic on both $H^7((C_u^5 \wedge C_r^{5,s})^{(9)}; \mathbb{Z}/2)$ and $H^7(Z'; \mathbb{Z}/2)$, which implies $y = 1$. Hence $Z' \simeq C_r^{9s} \vee C_s^9$. \square

From (16), (23), (24), together with Lemma 4.7, for $u \geq s, u \geq r$,

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C_r^{9,s} \vee C_r^{9,r} \vee M_{2^s}^4 \wedge C_\eta^5; \quad (28)$$

$$(C_u^5 \wedge C_r^{5,s})^{(9)} \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \vee C_r^{9,s}. \quad (29)$$

By (29), there is a cofibre sequence

$$S^9 \xrightarrow{\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}} M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \vee C_r^{9,s} \rightarrow C_u^5 \wedge C_r^{5,s}. \quad (30)$$

From (4), $\alpha = i_{\overline{C}}(t\varrho_6)$ for some $t \in \mathbb{Z}$. Hence $(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C_r^{9,r} \vee (C_s^9 \vee C_r^{9,s}) \cup \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \mathbf{C}S^9$. By (28),

$$(C_s^9 \vee C_r^{9,s}) \cup \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \mathbf{C}S^9 \simeq M_{2^s}^4 \wedge C_\eta^5 \vee C_r^{9,s} = (C_s^9 \vee C_r^{9,s}) \cup \begin{pmatrix} h_s \\ 0 \end{pmatrix} \mathbf{C}S^9.$$

By the proof of Lemma 2.7, there is a homotopy equivalence μ yielding following commutative diagram

$$\begin{array}{ccccccc} S^9 & \xrightarrow{\begin{pmatrix} \beta \\ \gamma \end{pmatrix}} & C_s^9 \vee C_r^{9,s} & \longrightarrow & (C_s^9 \vee C_r^{9,s}) \cup \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \mathbf{C}S^9 & \longrightarrow & S^{10} \\ \parallel & & \downarrow \mu' & & \downarrow \mu & & \parallel \\ S^9 & \xrightarrow{\begin{pmatrix} h_s \\ 0 \end{pmatrix}} & C_s^9 \vee C_r^{9,s} & \longrightarrow & (C_s^9 \vee C_r^{9,s}) \cup \begin{pmatrix} h_s \\ 0 \end{pmatrix} \mathbf{C}S^9 & \longrightarrow & S^{10} \end{array}$$

where μ' is the restriction of μ which is a self-homotopy equivalence of $C_s^9 \vee C_r^{9,s}$. So we get the commutative diagram

$$\begin{array}{ccc} S^9 & \xrightarrow{\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}} & M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \vee C_r^{9,s} \longrightarrow C_u^5 \wedge C_r^{5,s} \\ \parallel & & \downarrow \Gamma \\ S^9 & \xrightarrow{\begin{pmatrix} \alpha \\ h_s \\ 0 \end{pmatrix}} & M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \vee C_r^{9,s} \longrightarrow \mathbf{C} \begin{pmatrix} \alpha \\ h_s \end{pmatrix} \vee C_r^{9,s} \end{array}$$

$\begin{matrix} \vdots \\ \downarrow \tilde{\Gamma} \end{matrix}$

where $\Gamma = \begin{matrix} M_{2r}^3 \wedge C_\eta^5 \\ C_s^9 \vee C_r^{9,s} \end{matrix} \begin{matrix} M_{2r}^3 \wedge C_\eta^5 & C_s^9 \vee C_r^{9,s} \\ \hline 1 & 0 \\ 0 & \mu' \end{matrix}$ is a homotopy equivalence, which implies that $\tilde{\Gamma}$ is also a homotopy equivalence. Thus

$$C_u^5 \wedge C_r^{5,s} \simeq \mathbf{C} \begin{pmatrix} \alpha \\ h_s \end{pmatrix} \vee C_r^{9,s}, \quad \alpha = i_{\overline{C}}(t\varrho_6).$$

where $\mathbf{C} \begin{pmatrix} \alpha \\ h_s \end{pmatrix}$ is indecomposable (Corollary 4.4) which implies that $t = 1$ for $r = 1$ and $t \in \{1, 2\}$ for $r > 1$ (By (4)).

Lemma 4.8.

$$\mathbf{C} \begin{pmatrix} \alpha \\ h_s \end{pmatrix} \simeq C_\eta^5 \wedge C_r^{5,s}.$$

Proof. From Lemma 4.3, it suffices to show that $t = 1$, i.e., $\alpha = i_{\overline{C}}\varrho_6$.

For $r = 1$, $t = 1$.

For $r > 1$, by the similar computation of $\pi_9(C_\eta^5 \wedge C_r^{5,s})$ in Lemma 4.3 we get

$$\pi_9(\mathbf{C} \begin{pmatrix} \alpha \\ h_s \end{pmatrix}) \cong \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z}/2^s, & t = 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s+1}, & t = 1 \end{cases}. \quad (31)$$

t can be determined by computing

$$\pi_9(C_u^5 \wedge C_r^{5,s}) = \pi_9(\mathbf{C} \begin{pmatrix} \alpha \\ h_s \end{pmatrix}) \oplus \pi_9(C_r^{9,s}), \quad (32)$$

while $\pi_9(C_u^5 \wedge C_r^{5,s}) \cong [C^{14,u}, C_r^{13,s}] \cong [C^{7,u}, C_r^{6,s}]$. **Cof3** of $C^{7,u}$ yields the following exact sequence

$$[S^7, C_r^{6,s}] \xrightarrow{(2^u q_\eta)^*} [C_\eta^7, C_r^{6,s}] \rightarrow [C^{7,u}, C_r^{6,s}] \rightarrow [S^6, C_r^{6,s}] \xrightarrow{(2^u q_\eta)^*} [C_\eta^6, C_r^{6,s}]$$

From **Cof4** of $C_r^{k,s}$ and Lemma 3.1, for $k \geq 6$

$$[S^{k+1}, C_r^{k,s}] \cong [S^{k+1}, M_{2r}^{k-2}] \oplus [S^{k+1}, S^k] \cong \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, & r \geq 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, & r = 1 \end{cases}. \quad (33)$$

By the fact $u \geq r \geq 2$ and from (13), there is a short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s+1} \rightarrow [C^{7,u}, C_r^{6,s}] \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow 0.$$

Together with (32), we get $t = 1$. □

From Lemma 4.8, there is a decomposition

$$C_u^5 \wedge C_r^{5,s} \simeq C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s} \quad (u \geq s, u \geq r).$$

(v) For $u = s < r$, from (15) and (22), we have

$$\begin{aligned} (C_u^5 \wedge C_r^{5,s})/S^6 &\simeq C_r^{9,s} \vee C_\eta^{9,r} \vee M_{2^s}^4 \wedge C_\eta^5; \\ (C_u^5 \wedge C_r^{5,s})^{(9)} &\simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \vee C_r^{9,s}. \end{aligned}$$

By the same proof as in the case (iv), we get that

$$C_u^5 \wedge C_r^{5,s} \simeq C_s^{9,r} \vee C_\eta^5 \wedge C_s^{5,s} \quad (u = s < r).$$

In summary $C_u^5 \wedge C_r^{5,s}$ is

- ◇ homotopy equivalent to $C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s}$ for $u \geq s, u \geq r$;
- ◇ homotopy equivalent to $C_s^{9,r} \vee C_\eta^5 \wedge C_s^{5,s}$ for $u = s < r$;
- ◇ indecomposable, otherwise.

By $C^{5,u} \wedge C_r^{5,s} \simeq D(C_u^5 \wedge C_s^{5,r})$, we have $C^{5,u} \wedge C_r^{5,s}$ is

- ◇ homotopy equivalent to $C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s}$ for $u \geq s, u \geq r$;
- ◇ homotopy equivalent to $C_s^{9,r} \vee C_\eta^5 \wedge C_r^{5,r}$ for $u = r < r$;
- ◇ indecomposable, otherwise.

5 Decomposition of $C_r^{5,s} \wedge C_{r'}^{5,s'}$, $r, r', s, s' \in \mathbb{Z}^+$

5.1 Preliminaries

In this section, let u_3, u_4, \bar{u}_4, u_5 (resp. $u'_3, u'_4, \bar{u}'_4, u'_5$) be generators of $H^*(C_r^{5,s}; \mathbb{Z}/2)$ (resp. $H^*(C_{r'}^{5,s'}; \mathbb{Z}/2)$) which satisfy conditions (1) of Lemma 3.3.

$$H_*(C_r^{5,s} \wedge C_{r'}^{5,s'}) = \begin{cases} \mathbb{Z}/2^{m_{r,r'}}, & * = 6 \\ \mathbb{Z}/2^{m_{r,s'}} \oplus \mathbb{Z}/2^{m_{s,r'}} \oplus \mathbb{Z}/2^{m_{r,r'}}, & * = 7 \\ \mathbb{Z}/2^{m_{s,s'}} \oplus \mathbb{Z}/2^{m_{r,s'}} \oplus \mathbb{Z}/2^{m_{s,r'}}, & * = 8 \\ \mathbb{Z}/2^{m_{s,s'}}, & * = 9 \\ 0, & \text{otherwise} \end{cases}$$

$$H^*(C_r^{5,s} \wedge C_{r'}^{5,s'}; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\{u_3 \otimes u'_3\}, & * = 6 \\ \mathbb{Z}/2\{u_3 \otimes u'_4, u_3 \otimes \bar{u}'_4, u_4 \otimes u'_3, \bar{u}_4 \otimes u'_3\}, & * = 7 \\ \mathbb{Z}/2\left\{u_4 \otimes u'_4, u_4 \otimes \bar{u}'_4, \bar{u}_4 \otimes u'_4, \right. \\ \quad \left. \bar{u}_4 \otimes \bar{u}'_4, u_3 \otimes u'_5, u_5 \otimes u'_3\right\}, & * = 8 \\ \mathbb{Z}/2\{u_5 \otimes u'_4, u_5 \otimes \bar{u}'_4, u_4 \otimes u'_5, \bar{u}_4 \otimes u'_5\}, & * = 9 \\ \mathbb{Z}/2\{u_5 \otimes u'_5\}, & * = 10 \\ 0, & \text{otherwise} \end{cases}$$

The Steenrod operation action on $H^*(C_r^{5,s} \wedge C_{r'}^{5,s'}; \mathbb{Z}/2)$ is given as follows

- (i) $Sq^4(u_3 \otimes u'_3) = u_5 \otimes u'_5;$
- (ii) $Sq^2(u_3 \otimes u'_5) = Sq^2(u_5 \otimes u'_3) = u_5 \otimes u'_5;$
- (iii) $Sq^2(u_3 \otimes u'_3) = \begin{cases} u_3 \otimes u'_5 + u_4 \otimes u'_4 \otimes u_5 \otimes u'_3, & r = r' = 1 \\ u_3 \otimes u'_5 + u_5 \otimes u'_3, & \text{otherwise} \end{cases};$
- (iv) $Sq^2(u_3 \otimes u'_4) = u_5 \otimes u'_4; Sq^2(u_4 \otimes u'_3) = u_4 \otimes u'_5;$
 $Sq^2(u_3 \otimes \bar{u}'_4) = \begin{cases} u_5 \otimes \bar{u}'_4 + u_4 \otimes u'_5, & r = s' = 1 \\ u_5 \otimes \bar{u}'_4, & \text{otherwise} \end{cases};$
 $Sq^2(\bar{u}_4 \otimes u'_3) = \begin{cases} u_5 \otimes u'_4 + \bar{u}_4 \otimes u'_5, & r' = s = 1 \\ \bar{u}_4 \otimes u'_5, & \text{otherwise} \end{cases};$

Lemma 5.1. *If $C_r^{5,s} \wedge C_{r'}^{5,s'}$ is decomposable, then $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq X \vee C_l^{9,k}$ or $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq X \vee C_l^{9,k} \vee C_{l'}^{9,k'}$, where X is indecomposable, $H_6X \neq 0$ and $\{k, k'\} \subset \{m_{r,s'}, m_{s,r'}, m_{r,r'}\}$, $\{l, l'\} \subset \{m_{s,s'}, m_{r,s'}, m_{s,r'}\}$.*

Proof. Assume $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq X \vee Y$, X is indecomposable and $H_6X \neq 0$. From Lemma 2.6, $H_tX \cong H_t(C_r^{5,s} \wedge C_{r'}^{5,s'})$, $t = 6, 9$ and $\dim H_7X + \dim H_8X \geq 2$. It follows from the isomorphism $Sq^2 : H^7(C_r^{5,s} \wedge C_{r'}^{5,s'}; \mathbb{Z}/2) \xrightarrow{\cong} H^9(C_r^{5,s} \wedge C_{r'}^{5,s'}; \mathbb{Z}/2)$ that Moore spaces are not summands of Y . Hence there will be following two cases

- (i) $\dim H_7X + \dim H_8X = 2$ which implies $\dim H_7X = \dim H_8X = 1$. Note that $Y \in \mathbf{A}_n^2$. Thus $Y \simeq C_l^{9,k} \vee C_{l'}^{9,k'}$ for some $\{k, k'\} \subset \{m_{r,s'}, m_{s,r'}, m_{r,r'}\}$, $\{l, l'\} \subset \{m_{s,s'}, m_{r,s'}, m_{s,r'}\}$.
- (ii) $\dim H_7X + \dim H_8X = 4$ which implies $\dim H_7X = \dim H_8X = 2$ and $Y \simeq C_l^{9,k}$ for some $k \in \{m_{r,s'}, m_{s,r'}, m_{r,r'}\}$, $l \in \{m_{s,s'}, m_{r,s'}, m_{s,r'}\}$.

□

Let $\max = \max\{r, s, r', s'\}$. By $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq D(C_s^{5,r} \wedge C_{s'}^{5,r'})$, we can assume $\max = s$.

Lemma 5.2. *If $\max = s > r', s'$, then $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_{r'}^{9,s'} \vee C_r^{5,s} \wedge C_{r'}^{5,s'}$, hence*

$$C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq \begin{cases} C_{r'}^{9,s'} \vee C_{r'}^{9,s'} \vee C_\eta^{5,s} \wedge C_{r'}^{5,s'}, & s > r', s' \text{ and } r \geq r', s' \\ C_{r'}^{9,s'} \vee C_{s'}^{9,r'} \vee C_\eta^{5,s} \wedge C_r^{5,r}, & s > r' > s' = r \\ C_{r'}^{9,s'} \vee C_r^{5,s} \wedge C_{r'}^{5,s'}, & s > r', s' \text{ and } s' \neq r < r' \text{ or } r < s' \end{cases}.$$

Proof. From **Cof3** of $C_r^{5,s}$ and $|(C_{r'}^{5,s'}, C_{r'}^{5,s'})| = \max\{2^{s'+1}, 2^{r'+1}\} [1]$, we get

$$\begin{array}{ccccc} S^4 \wedge C_{r'}^{5,s'} & \xrightarrow{\left(\begin{smallmatrix} i\eta \\ 2^s \end{smallmatrix}\right) \wedge 1} & (M_{2r}^3 \vee S^3) \wedge C_{r'}^{5,s'} & \longrightarrow & C_r^{5,s} \wedge C_{r'}^{5,s'} \\ & \searrow & \uparrow \simeq & & \\ & \left(\begin{smallmatrix} i\eta \wedge 1 \\ 2^s \wedge 1 = 0 \end{smallmatrix}\right) & M_{2r}^3 \vee C_{r'}^{5,s'} \wedge S^4 \vee C_{r'}^{5,s'} & & \end{array}$$

Thus $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_{r'}^{5,s'} \vee (M_{2r}^3 \vee C_{r'}^{5,s'}) \cup_{i\eta \wedge 1} \mathbf{C}(S^4 \vee C_{r'}^{5,s'}) \simeq C_{r'}^{5,s'} \vee C_r^5 \wedge C_{r'}^{5,s'}$. By the decomposability of $C_r^5 \wedge C_{r'}^{5,s'}$ in Subsection 4.3, the Lemma is obtained. \square

Now the following cases remain:

(I) $\max = s = r'$

(i) $s = r' > s' > r$; (ii) $s = r' > s' = r$; (iii) $s = r' = s' > r$; (iv) $s = r' = s' = r$;
(v) $s = r' > r > s'$; (vi) $s = r' = r > s'$;

(II) $\max = s = s'$

(i) $s = s' > r' > r$; (ii) $s = s' > r > r'$; (iii) $s = s' > r' = r$; (iv) $s = s' = r > r'$.

By $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq D(C_s^{5,r} \wedge C_{s'}^{5,r'}) \simeq D(C_{s'}^{5,r'} \wedge C_s^{5,r})$, case (I)(iii) is dual to case (I)(vi); case (I)(i) is dual to case (I)(v).

By $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_{r'}^{5,s'} \wedge C_r^{5,s}$, case (II)(i) is the same as the case (II)(ii); case (II)(iv) is the same as the case (I)(iii).

Hence it suffices to compute the following cases, denoted by **Cases** \star :

(a) $s = r' > s' > r$; (b) $s = r' > s' = r$; (c) $s = r' = s' > r$;

(d) $s = r' = s' = r$; (e) $s = s' > r' > r$; (f) $s = s' > r' = r$.

We will prove the (a) of **Cases** \star and omit the proofs of other cases since they are similar or easier.

5.2 $(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6$ and $(C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)}$ for $s = r' > s' > r$

(1) Determining $(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6$

For $S_1^3 \vee M_{2s}^3 \xrightarrow{f=(2^r, \eta q)} S_a^3 \rightarrow C_r^{5,s}$ and $S_2^3 \vee M_{2s'}^3 \xrightarrow{f'=(2^{r'}, \eta q)} S_b^3 \rightarrow C_{r'}^{5,s'}$.

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6 \simeq (\Sigma(S_1^3 \vee M_{2s}^3) \wedge S_b^3 \vee \Sigma S_a^3 \wedge (S_2^3 \vee M_{2s'}^3)) \cup_{\mathcal{A}} \mathbf{C}\Sigma(S_1^3 \vee M_{2s}^3) \wedge (S_2^3 \vee M_{2s'}^3),$$

where $\mathcal{A} = \begin{pmatrix} \Sigma 1 \wedge f' \\ -\Sigma f \wedge 1 \end{pmatrix}$, i.e.,

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6 \simeq (S^7 \vee M_{2s}^7 \vee S^7 \vee M_{2s'}^7) \cup_{\mathcal{A}} \mathbf{C}(S^7 \vee M_{2s}^7 \vee M_{2s'}^7 \vee \Sigma M_{2s}^3 \wedge M_{2s'}^3).$$

where $\mathcal{A} =$

S^7	$M_{2s'}^7$	M_{2s}^7	$\Sigma M_{2s}^3 \wedge M_{2s'}^3$
$\Sigma M_{2s}^3 \wedge S_b^3 = M_{2s}^7$	$2^{r'}$	ηq	0
S^7	0	0	$2^{r'}$
$\Sigma S_a^3 \wedge M_{2s'}^3 = M_{2s'}^7$	-2^r	0	ηq
	0	-2^r	0
	0	0	$\Sigma \eta q \wedge 1$

With identification $\Sigma M_{2^s}^3 \wedge M_{2^{s'}}^3 \simeq M_{2^{s'}}^7 \vee M_{2^{s'}}^8$,

$$\mathcal{A} = \begin{array}{c} S^7 \\ M_{2^s}^7 \\ S^7 \\ M_{2^{s'}}^7 \end{array} \begin{array}{ccccc} S^7 & M_{2^{s'}}^7 & M_{2^s}^7 & M_{2^{s'}}^7 & M_{2^{s'}}^8 \\ \hline 2^{r'} & \eta q & 0 & 0 & 0 \\ 0 & 0 & 0 & i\eta q & \xi_s^{s'} + \kappa i\eta\eta q \\ -2^r & 0 & \eta q & 0 & 0 \\ 0 & -2^r & 0 & 0 & \eta \wedge 1 \end{array}$$

$$\xrightarrow{\cong} \begin{array}{c} S^7 \\ M_{2^s}^7 \\ S^7 \\ M_{2^{s'}}^7 \end{array} \begin{array}{ccccc} S^7 & M_{2^{s'}}^7 & M_{2^s}^7 & M_{2^{s'}}^7 & M_{2^{s'}}^8 \\ \hline 0 & \eta q & 0 & 0 & 0 \\ 0 & 0 & 0 & i\eta q & \xi_s^{s'} + \kappa i\eta\eta q \\ 2^r & 0 & \eta q & 0 & 0 \\ 0 & 2^r & 0 & 0 & \eta \wedge 1 \end{array},$$

Thus $(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6 \simeq C_r^{9,s} \vee L$, where L is the mapping cone of the map

$$\begin{array}{c}
M_{2s'}^7 \quad M_{2s'}^7 \quad M_{2s'}^8 \\
\begin{array}{ccc}
S^7 & \eta q & 0 \\
M_{2s}^7 & 0 & i\eta q \\
M_{2s'}^7 & 2^r & 0
\end{array}
\end{array}
\begin{array}{c}
0 \\
\xi_{s'} + \kappa i\eta\eta q \\
\eta \wedge 1
\end{array}
.$$

Lemma 5.3. $L \simeq C^{9,r} \vee (C_s^{9,s'} \vee C_r^9) \cup_{\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}} \mathbf{CS}^9$ where $\alpha = i_{\underline{M}} \begin{pmatrix} \xi_s \\ 0 \end{pmatrix}$ and γ is determined by $q_S \gamma = \begin{pmatrix} 0 \\ 2_{s'} \end{pmatrix}$, i.e., there are commutative diagrams

$$\begin{array}{ccc}
\begin{pmatrix} \xi_s \\ 0 \end{pmatrix} & \begin{array}{c} S^9 \\ \downarrow \alpha \end{array} & \\
\swarrow & & \searrow \\
M_{2^s}^7 \vee S^8 & \xrightarrow{i_M} & C_{s,s'}^{9,s'};
\end{array}
\qquad
\begin{array}{ccc}
S^9 & \begin{pmatrix} 0 \\ 2^{s'} \end{pmatrix} & \\
\downarrow \gamma & & \searrow \\
C_r^9 & \xrightarrow{q_S} & S^8 \vee S^9.
\end{array}$$

Proof. By the compositions

$$S^8 \xrightarrow{i} M_{2^{s'}}^8 \xrightarrow{\eta \wedge 1} M_{2^{s'}}^7 \quad \text{and} \quad S^8 \xrightarrow{i} M_{2^{s'}}^8 \xrightarrow{\xi_s' + \kappa i \eta \eta q} M_{2^s}^7,$$

$L^{(9)}$ is the mapping cone of the map $\begin{array}{c} S^7 \\ M_{2^{s'}}^7 \\ M_{2^{s'}}^7 \end{array} \begin{array}{|c|c|c|} \hline M_{2^{s'}}^7 & M_{2^{s'}}^7 & S^8 \\ \hline \eta q & 0 & 0 \\ 0 & i\eta q & 0 \\ 2^r & 0 & i\eta \\ \hline \end{array}$. Hence $L^{(9)} \simeq C_s^{9,s'} \vee L_1$,

where L_1 is the mapping cone of $\begin{array}{cc} S^7 & M_{2s'}^7 \\ M_{2s'}^7 & \begin{bmatrix} \eta q & 0 \\ 2^r & i\eta \end{bmatrix} \end{array}$. Let W_1 be the mapping cone

of the map $\begin{array}{cc} S^7 & S^8 \\ M_{2^{s'}}^7 & \begin{bmatrix} 0 & 0 \\ 2^r i_7 & i_\eta \end{bmatrix} \end{array}$, i.e., $W_1 \simeq (S^7 \vee M_{2^{s'}}^7) \cup \begin{pmatrix} 0 & 0 \\ 2^r i_7 & i_\eta \end{pmatrix} \mathbf{C}(S^7 \vee S^8) \simeq S^7 \vee M_{2^{s'}}^7 \cup_{(2^r i, i_\eta)} \mathbf{C}(S^7 \vee S^8) \simeq S^7 \vee S^8 \vee C_r^9$ (by (26)).

Similarly as in the proof of Lemma 4.7, there is a cofibre sequence

$$S^8 \xrightarrow{\begin{pmatrix} x\eta \\ \hat{\beta} \\ \hat{\gamma} \end{pmatrix}} W_1 \simeq S^7 \vee S^8 \vee C_r^9 \rightarrow L_1 \rightarrow S^9,$$

where $x \in \{0, 1\}$. From $H_8 L_1 = \mathbb{Z}/2^r$ and $\pi_8 C_r^9 = 0$ we get $\hat{\beta} = 2^r$ and $\hat{\gamma} = 0$. By the isomorphisms $H^7((C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6; \mathbb{Z}/2) \xrightarrow{\cong} H^7(C_r^{5,s} \wedge C_{r'}^{5,s'}; \mathbb{Z}/2)$ and $H^9(C_r^{5,s} \wedge C_{r'}^{5,s'}; \mathbb{Z}/2) \xrightarrow{\cong} H^9((C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)}; \mathbb{Z}/2)$ induced by canonical pinch map and canonical inclusion respectively, Sq^2 on $H^7(L; \mathbb{Z}/2)$ and $H^7(L_1; \mathbb{Z}/2)$ are also isomorphic, which implies that $x = 1$. So $L_1 \simeq C_r^{9,r} \vee C_r^9$ and $L^{(9)} \simeq C_s^{9,s'} \vee C_r^{9,r} \vee C_r^9$. There is a cofibre sequence

$$S^9 \xrightarrow{f_L = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}} C_s^{9,s'} \vee C_r^{9,r} \vee C_r^9 \rightarrow L \rightarrow S^{10} \rightarrow C_s^{10,s'} \vee C_r^{10,r} \vee C_r^{10}.$$

$\alpha = i_{\underline{M}} \begin{pmatrix} z\xi_s \\ x\eta \end{pmatrix}$, $\beta = i_S \begin{pmatrix} 0 \\ y\eta \end{pmatrix}$, γ is determined by $q_S \gamma = \begin{pmatrix} w\eta \\ a \end{pmatrix}$, where $x, y, z, w \in \{0, 1\}$ and $a \in \mathbb{Z}$.

$$\begin{array}{ccc} \begin{pmatrix} z\xi_s \\ x\eta \end{pmatrix} & \swarrow & S^9 \\ & \searrow \alpha & \downarrow \\ M_{2^s}^7 \vee S^8 & \xrightarrow{i_{\underline{M}}} & C_s^{9,s'} \end{array}; \quad \begin{array}{ccc} \begin{pmatrix} 0 \\ y\eta \end{pmatrix} & \swarrow & S^9 \\ & \searrow \beta & \downarrow \\ S^7 \vee S^8 & \xrightarrow{i_S} & C_r^{9,r} \end{array}; \quad \begin{array}{ccc} S^9 & & \begin{pmatrix} w\eta \\ a \end{pmatrix} \\ \gamma \downarrow & \searrow & \\ C_r^9 & \xrightarrow{q_S} & S^8 \vee S^9 \end{array}.$$

$a = 2^{s'}$ from $[L, S^{10}] \cong \mathbb{Z}/2^{s'}$. From Proposition 2.1, for $s = r' > s' > r$

$$[(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6, S^8] \cong [C_r^{5,s} \wedge C_{r'}^{5,s'}, S^8] \cong [C_r^{5,s}, C_{s'}^{5,r'}] \cong \mathbb{Z}/2^s \oplus \mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^{r+1}.$$

Together with $[(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6, S^8] \cong [C_r^{9,s}, S^8] \oplus [L, S^8]$, we get

$$[L, S^8] \cong \mathbb{Z}/2^s \oplus \mathbb{Z}/2^{r+1}.$$

On the other hand, by the cofibre sequence of L above, there is an exact sequence

$$0 \rightarrow \frac{\mathbb{Z}/2}{\langle x, y, z, w \rangle} \rightarrow [L, S^8] \rightarrow \mathbb{Z}/2^{s+1} \oplus \mathbb{Z}/2^{r+1} \xrightarrow{(\alpha^*, \gamma^*)} \mathbb{Z}/2$$

where $\alpha^*(1) = z, \gamma^*(1) = w$. Hence $w = 0, z = 1$. By the following commutative diagram

$$\begin{array}{ccccc} S^9 & \xrightarrow{f_L} & C_s^{9,s'} \vee C_r^{9,r} \vee C_r^9 & \longrightarrow & L \\ \parallel & & \downarrow \Theta & & \downarrow \hat{\Theta} \\ S^9 & \xrightarrow{\theta = \Theta f_L} & C_s^{9,s'} \vee C_r^{9,r} \vee C_r^9 & \longrightarrow & \mathbf{C}_\theta \end{array}$$

where $\Theta = \begin{matrix} & C_s^{9,s'} & C_r^{9,r} & C_r^9 \\ \begin{matrix} C_s^{9,s'} \\ C_r^{9,r} \\ C_r^9 \end{matrix} & \begin{bmatrix} \hat{\mu} & 0 & 0 \\ \hat{\lambda} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$ is a homotopy equivalence, $\hat{\mu}$ and $\hat{\lambda}$ are induced by the following commutative diagrams

$$\begin{array}{ccc} S^8 & \xrightarrow{\begin{pmatrix} i\eta \\ 2^{s'} \end{pmatrix}} M_{2^s}^7 \vee S^8 & \xrightarrow{i_M} C_s^{9,s'} \\ \parallel & \downarrow \begin{pmatrix} 1, 0 \\ xq, 1 \end{pmatrix} & \downarrow \hat{\mu} \\ S^8 & \xrightarrow{\begin{pmatrix} i\eta \\ 2^{s'} \end{pmatrix}} M_{2^s}^7 \vee S^8 & \xrightarrow{i_M} C_s^{9,s'} \end{array}, \quad \begin{array}{ccc} S^8 & \xrightarrow{\begin{pmatrix} i\eta \\ 2^{s'} \end{pmatrix}} M_{2^s}^7 \vee S^8 & \xrightarrow{i_M} C_s^{9,s'} \\ \downarrow 0 & \downarrow \begin{pmatrix} 0, 0 \\ yq, 0 \end{pmatrix} & \downarrow \hat{\lambda} \\ S^8 & \xrightarrow{\begin{pmatrix} \eta \\ 2^r \end{pmatrix}} S^7 \vee S^8 & \xrightarrow{i_S} C_r^{9,r} \end{array}.$$

Then $\theta = \Theta f_L = \begin{pmatrix} \hat{\mu}\alpha \\ \hat{\lambda}\alpha + \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \hat{\mu}\alpha \\ 0 \\ \gamma \end{pmatrix}$. Rewrite $\hat{\mu}\alpha$ as α ,

$$L \simeq \mathbf{C}_\theta \simeq C_r^{9,r} \vee (C_s^{9,s'} \vee C_r^9) \cup \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \mathbf{C}S^9$$

α, γ satisfy the conditions in the Lemma. \square

Thus

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6 \simeq C_r^{9,s} \vee C_r^{9,r} \vee (C_s^{9,s'} \vee C_r^9) \cup \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \mathbf{C}S^9, \quad s = r' > s' > r. \quad (34)$$

(2) Determining $(C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)}$

$$\begin{aligned} S_1^4 &\xrightarrow{g=\begin{pmatrix} i\eta \\ 2^s \end{pmatrix}} M_{2^r}^3 \vee S_a^4 \rightarrow C_r^{5,s}; \quad S_2^4 \xrightarrow{g'=\begin{pmatrix} i\eta \\ 2^{s'} \end{pmatrix}} M_{2^{r'}}^3 \vee S_b^4 \rightarrow C_{r'}^{5,s'}, \\ (C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)} &\simeq (M_{2^r}^3 \vee S_a^4) \wedge (M_{2^{r'}}^3 \vee S_b^4) \cup_{\mathcal{B}} \mathbf{C}(S_1^4 \wedge (M_{2^{r'}}^3 \vee S_b^4) \vee (M_{2^r}^3 \vee S_a^4) \wedge S_2^4), \end{aligned}$$

where $\mathcal{B} = (g \wedge 1, 1 \wedge g')$, i.e.,

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)} \simeq (M_{2^r}^3 \wedge M_{2^{r'}}^3 \vee M_{2^r}^7 \vee M_{2^{r'}}^7 \vee S^8) \cup_{\mathcal{B}} \mathbf{C}(M_{2^r}^7 \vee S^8 \vee M_{2^{r'}}^7 \vee S^8),$$

$$\mathcal{B} = \begin{matrix} M_{2^r}^3 \wedge M_{2^{r'}}^3 \\ M_{2^r}^7 \\ M_{2^{r'}}^7 \\ S^8 \end{matrix} \begin{bmatrix} M_{2^r}^7 & S^8 & M_{2^{r'}}^7 & S^8 \\ 1 \wedge i\eta & 0 & i\eta \wedge 1 & 0 \\ 2^{s'} & 0 & 0 & i\eta \\ 0 & i\eta & 2^s & 0 \\ 0 & 2^{s'} & 0 & 2^s \end{bmatrix} \cong \begin{matrix} M_{2^r}^3 \wedge M_{2^{r'}}^3 \\ M_{2^r}^7 \\ M_{2^{r'}}^7 \\ S^8 \end{matrix} \begin{bmatrix} M_{2^r}^7 & S^8 & M_{2^{r'}}^7 & S^8 \\ 1 \wedge i\eta & 0 & i\eta \wedge 1 & 0 \\ 0 & 0 & 0 & i\eta \\ 0 & i\eta & 0 & 0 \\ 0 & 2^{s'} & 0 & 0 \end{bmatrix}$$

by noting that $2^{s'} = 0 \in [M_{2^r}^7, M_{2^r}^7]$ and $2^s = 0 \in [M_{2^{r'}}^7, M_{2^{r'}}^7]$ for $s = r' > s' > r$. From (19), for $r' > r$, $(M_{2^r}^3 \wedge M_{2^{r'}}^3) \cup_{(1 \wedge i\eta, i\eta \wedge 1)} \mathbf{C}(M_{2^r}^7 \vee M_{2^{r'}}^7) \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_r^{9,r'}$. Thus

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)} \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_r^{9,r'} \vee C_{r'}^{9,s'} \vee C_r^9 \quad \text{for } s = r' > s' > r \quad (35)$$

5.3 Decomposition of $C_r^{5,s} \wedge C_{r'}^{5,s'}$ for $s = r' > s' > r$

Denote column vector by $(\zeta_1, \zeta_2, \dots, \zeta_s)^T$ in the rest of the paper.

From (35), there is a cofibre sequence

$$S^9 \xrightarrow{(\delta_1, \mu, \nu, \omega)^T} M_{2r}^3 \wedge C_\eta^5 \vee C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9 \rightarrow C_r^{5,s} \wedge C_{r'}^{5,s'}$$

Since $\delta_1 = i_{\overline{C}} t_1 \varrho_6$ for some $t_1 \in \mathbb{Z}$,

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6 \simeq C_r^{9,r} \vee (C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9) \cup_{(\mu, \nu, \omega)^T} \mathbf{C} S^9.$$

On the other hand, from (34)

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6 \simeq C_r^{9,r} \vee (C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9) \cup_{(0, \alpha, \gamma)^T} \mathbf{C} S^9.$$

Thus $(C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9) \cup_{(\mu, \nu, \omega)^T} \mathbf{C} S^9 \simeq (C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9) \cup_{(0, \alpha, \gamma)^T} \mathbf{C} S^9$, which are 2-local spaces. Consequently, there is a homotopy equivalence λ yielding the following homotopy commutative diagram

$$\begin{array}{ccccccc} S^9 & \xrightarrow{(\mu, \nu, \omega)^T} & C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9 & \longrightarrow & \mathbf{C}_{(\mu, \nu, \omega)^T} & \longrightarrow & S^{10} \\ \parallel & & \downarrow \lambda' & & \downarrow \lambda \simeq & & \parallel \\ S^9 & \xrightarrow{(0, \alpha, \gamma)^T} & C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9 & \longrightarrow & \mathbf{C}_{(0, \alpha, \gamma)^T} & \longrightarrow & S^{10} \end{array}$$

where λ' is the restriction of λ , which is a self-homotopy equivalence of $C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9$.

Let $\tilde{\Gamma}$ be induced by the following commutative diagram

$$\begin{array}{ccccccc} S^9 & \xrightarrow{(\delta_1, \mu, \nu, \omega)^T} & M_{2r}^3 \wedge C_\eta^5 \vee C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9 & \longrightarrow & C_r^{5,s} \wedge C_{r'}^{5,s'} & & \\ \parallel & & \downarrow \Gamma & & \downarrow \tilde{\Gamma} & & \\ S^9 & \xrightarrow{\theta = \Gamma(\delta_1, \mu, \nu, \omega)^T} & M_{2r}^3 \wedge C_\eta^5 \vee C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9 & \longrightarrow & \mathbf{C}_\theta & & \end{array}$$

where $\Gamma = \begin{array}{c} M_{2r}^3 \wedge C_\eta^5 \\ C_r^{9,s} \vee C_{r'}^{9,s'} \vee C_r^9 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & \lambda' \end{bmatrix}$ is a self-homotopy equivalence, then $\tilde{\Gamma}$ is also a homotopy equivalence.

$$\theta = \Gamma(\delta_1, \mu, \nu, \omega)^T = (\delta_1, \lambda'(\mu, \nu, \omega)^T)^T = (\delta_1, 0, \alpha, \gamma)^T.$$

Consequently, for $s = r' > s' > r$,

$$C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq \mathbf{C}_\theta \simeq C_r^{9,s} \vee Q_1, \quad Q_1 = (M_{2r}^3 \wedge C_\eta^5 \vee C_{r'}^{9,s'} \vee C_r^9) \cup_{(\delta_1, \alpha, \gamma)^T} \mathbf{C} S^9.$$

Lemma 5.4. $Q_1 \simeq C_r^{5,s'} \wedge C_r^{5,s}$, $s = r' > s' > r$.

Proof. (1) **Determining δ_1 , i.e., t_1**

By Lemma 5.1, $C_r^{5,s} \wedge C_{r'}^{5,s'}$ can not split out $M_{2r}^3 \wedge C_\eta^5$, implies that $\delta_1 \neq 0$ in $\pi_9(M_{2r}^3 \wedge C_\eta^5)$. Hence we can assume that $t_1 = 1$ for $r = 1$ and $t_1 \in \{1, 2\}$ for $r \geq 2$. Next we determine t_1 for $r \geq 2$.

Lemma 5.5. For $s \geq r', s'$ and $r \geq 2$, there is a short exact sequence of abelian groups

$$0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+\delta_{r'}} \rightarrow \pi_9(C_r^{5,s} \wedge C_{r'}^{5,s'}) \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow 0.$$

where $\delta_{r'} = 0$ for $r' = 1$ and $\delta_{r'} = 1$ for $r' > 1$.

Proof. We only prove the case $r' > 1$.

By Proposition 2.1,

$$\pi_9(C_r^{5,s} \wedge C_{r'}^{5,s'}) \cong [C_s^{5,r} \wedge C_{s'}^{5,r'}, S^7] \cong [C_s^{7,r}, C_{r'}^{6,s'}].$$

By **Cof5** of $C_s^{7,r}$, there is an exact sequence

$$[S^7, C_{r'}^{6,s'}] \xrightarrow{(2^r p_1 q_S)^*} [C_s^7, C_{r'}^{6,s'}] \rightarrow [C_s^{7,r}, C_{r'}^{6,s'}] \rightarrow [S^6, C_{r'}^{6,s'}] \xrightarrow{0} [C_s^7, C_{r'}^{6,s'}]. \quad (36)$$

From (33), $(2^r p_1 q_S)^*$ is zero for $r \geq 2$. For $s \geq r', s'$, $[C_s^7, C_{r'}^{6,s'}] \cong [S^9, C_r^{5,s} \wedge C_{r'}^{5,s'}] \cong \pi_9(C_{r'}^{9,s'} \vee C_\eta^5 \wedge C_{r'}^{5,s'}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}$ (the last isomorphism is from (13)). \square

There is a cofibre sequence

$$S^9 \xrightarrow{(\delta_1, \alpha_*, \gamma_*)^T} M_{2r}^3 \wedge C_\eta^5 \vee C_{r'}^{9,s'} \vee C_r^9 \rightarrow Q_1 \rightarrow S^{10}. \quad (37)$$

which implies following exact sequence

$$\mathbb{Z} \xrightarrow{(\delta_{1*}, \alpha_*, \gamma_*)^T} \mathbb{Z}/4 \langle i_{\overline{C}} \varrho_6 \rangle \oplus \mathbb{Z}/2 \langle i_{\underline{M}} j_1 \xi_{r'} \rangle \oplus \mathbb{Z}/2 \langle i_{\underline{M}} j_2 \eta \rangle \oplus \mathbb{Z}/2 \langle (j_1 \eta)_S^- \rangle \oplus \mathbb{Z} \langle (2j_2)_S^- \rangle \rightarrow \pi_9 Q_1 \rightarrow 0$$

where $\delta_{1*}(1) = t_1 i_{\overline{C}} \varrho_6$, $\alpha_*(1) = i_{\underline{M}} j_1 \xi_{r'}$ and $\gamma_*(1) = 2^{s'-1} (2j_2)_S^-$.

$$\pi_9 Q_1 \cong \frac{\mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (t_1, 1, 0, 0, 2^{s'-1}) \rangle} \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}, & t_1 = 1 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'}, & t_1 = 2 \end{cases}. \quad (38)$$

Since $\pi_9(C_r^{5,s} \wedge C_{r'}^{5,s'}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \pi_9 Q_1$ induced by $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_r^{9,s} \vee Q_1$ for $s = r' > s' > r$, together with the short exact sequence for $\pi_9(C_r^{5,s} \wedge C_{r'}^{5,s'})$ in Lemma 5.5, we have

$$\pi_9 Q_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}, \quad \text{i.e., } t_1 = 1, \quad \delta_1 = i_{\overline{C}} \varrho_6.$$

(2) The cell structure of $C^{5,s'} \wedge C_r^{5,s}$

Apply Lemma 2.4 to **Cof1** of $C^{5,s'}$ and **Cof4** of $C_r^{5,s}$ to get

$$(C^{5,s'} \wedge C_r^{5,s})^{(9)} \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_{r'}^{9,s'} \vee C_r^9.$$

There is a cofibre sequence

$$S^9 \xrightarrow{(\hat{\delta}, \hat{\alpha}, \hat{\gamma})^T} M_{2^r}^3 \wedge C_\eta^5 \vee C_{r'}^{9,s'} \vee C_r^9 \rightarrow C^{5,s'} \wedge C_r^{5,s} \rightarrow S^{10} \quad (39)$$

where $\hat{\delta} = i_{\overline{C}} \hat{t} \ell_6$, $\hat{\alpha} = i_{\underline{M}} \begin{pmatrix} \hat{x} \xi_{r'} \\ \hat{y} \eta \end{pmatrix}$ and $\hat{\gamma}$ is determined by $q_S \begin{pmatrix} \hat{z} \eta \\ \hat{a} \end{pmatrix}$ for $\hat{x}, \hat{y}, \hat{z} \in \{0, 1\}$ and $\hat{t}, \hat{a} \in \mathbb{Z}$, i.e.,

$$\begin{array}{ccccc} S^9 & & S^9 & & S^9 \\ \swarrow \hat{t} \ell_6 & \downarrow \hat{\delta} & \swarrow \begin{pmatrix} \hat{x} \xi_{r'} \\ \hat{y} \eta \end{pmatrix} & \downarrow \hat{\alpha} & \swarrow \begin{pmatrix} \hat{z} \eta \\ \hat{a} \end{pmatrix} \\ S^6 & \xrightarrow{i_{\overline{C}}} M_{2^r}^3 \wedge C_\eta^5 & ; & M_{2^{r'}}^7 \vee S^8 \xrightarrow{i_{\underline{M}}} C_{r'}^{9,s'} & ; & C_r^9 \xrightarrow{q_S} S^8 \vee S^9 \end{array}$$

$\hat{a} = 2^{s'}$ for $H_9(C^{5,s'} \wedge C_r^{5,s}) \cong \mathbb{Z}/2^{s'}$. Cofibre sequence (39) induces

$$0 \rightarrow \frac{\mathbb{Z}/2}{\langle \hat{x}, \hat{y}, \hat{z} \rangle} \rightarrow [C^{5,s'} \wedge C_r^{5,s}, S^8] \rightarrow \mathbb{Z}/2^{r'+1} \oplus \mathbb{Z}/2^{r+1} \xrightarrow{(\hat{\alpha}^*, \hat{\gamma}^*)} \mathbb{Z}/2$$

where $\hat{\alpha}^*(1) = \hat{x}$ and $\hat{\gamma}^*(1) = \hat{z}$. Since $C^{5,s'} \wedge C_r^{5,s}$ is indecomposable for $s = r' > s' > r$, $\hat{x} = 1$ or $\hat{y} = 1$, which implies $\frac{\mathbb{Z}/2}{\langle \hat{x}, \hat{y}, \hat{z} \rangle} = 0$. On the other hand,

$$[C^{5,s'} \wedge C_r^{5,s}, S^8] \cong [C_r^{5,s}, C_{s'}^5] \cong \mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^{r'}.$$

Thus $\hat{x} = 1$ and $\hat{z} = 0$. Now from the following commutative diagram

$$\begin{array}{ccccc} S^9 & \xrightarrow{(\hat{\delta}, \hat{\alpha}, \hat{\gamma})^T} & M_{2^r}^3 \wedge C_\eta^5 \vee C_{r'}^{9,s'} \vee C_r^9 & \longrightarrow & C^{5,s'} \wedge C_r^{5,s} \\ \parallel & & \downarrow \Theta (\simeq) & & \downarrow \tilde{\Theta} (\simeq) \\ S^9 & \xrightarrow{\Theta (\hat{\delta}, \hat{\alpha}, \hat{\gamma})^T} & M_{2^r}^3 \wedge C_\eta^5 \vee C_{r'}^{9,s'} \vee C_r^9 & \longrightarrow & \mathbf{C}_\theta \end{array}$$

where $\Theta = \begin{matrix} M_{2^r}^3 \wedge C_\eta^5 \\ C_{r'}^{9,s'} \\ C_r^9 \end{matrix} \begin{bmatrix} M_{2^r}^3 \wedge C_\eta^5 & C_{r'}^{9,s'} & C_r^9 \\ 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a self-homotopy equivalence and λ is induced by

$$\begin{array}{ccccc} S^8 & \xrightarrow{\begin{pmatrix} i\eta \\ 2^{s'} \end{pmatrix}} & M_{2^{r'}}^7 \vee S^8 & \xrightarrow{i_{\underline{M}}} & C_{r'}^{9,s'} \\ \parallel & & \downarrow \begin{pmatrix} 1, 0 \\ \hat{y}q, 1 \end{pmatrix} & & \downarrow \lambda \\ S^8 & \xrightarrow{\begin{pmatrix} i\eta \\ 2^{s'} \end{pmatrix}} & M_{2^{r'}}^7 \vee S^8 & \xrightarrow{i_{\underline{M}}} & C_{r'}^{9,s'} \end{array},$$

we can assume $\hat{y} = 0$. Next we are going to calculate $\hat{\delta}$, i.e., \hat{t} .

$\hat{t} = 1$ for $r = 1$ and $\hat{t} \in \{1, 2\}$ for $r \geq 2$ since $\hat{\delta} \neq 0$.

For the case $r \geq 2$, similarly as the calculation of $\pi_9 Q_1$, we get

$$\pi_9(C^{5,s'} \wedge C_r^{5,s}) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}, & \hat{t} = 1 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'}, & \hat{t} = 2 \end{cases}. \quad (40)$$

On the other hand, $\pi_9(C^{5,s'} \wedge C_r^{5,s}) \cong [C_{s'}^7, C_r^{6,s}]$ and from **Cof3** of $C_{s'}^7$, we get the following exact sequence

$$[C_\eta^8, C_r^{6,s}] \xrightarrow{(i_\eta 2^{s'})^* = 0} [S^6, C_r^{6,s}] \rightarrow [C_{s'}^7, C_r^{6,s}] \rightarrow [C_\eta^7, C_r^{6,s}] \xrightarrow{(i_\eta 2^{s'})^*} [S^5, C_r^{6,s}].$$

There is a commutative diagram for the morphism $(i_\eta 2^{s'})^*$

$$\begin{array}{ccccc} & & \xrightarrow{(i_\eta 2^{s'})^*} & & \\ & \swarrow & & \searrow & \\ [C_\eta^7, C_r^{6,s}] & \xrightarrow{i_\eta^*} & [S^5, C_r^{6,s}] & \xrightarrow{(2^{s'})^*} & [S^5, C_r^{6,s}] \\ \uparrow (i_M)^* & & \uparrow (i_M)^* & & \uparrow (i_M)^* \\ [C_\eta^7, M_{2^r}^4 \vee S^5] & \xrightarrow{i_\eta^*} & [S^5, M_{2^r}^4 \vee S^5] & \xrightarrow{(2^{s'})^*} & [S^5, M_{2^r}^4 \vee S^5] \\ \uparrow \begin{pmatrix} i_\eta \\ 2^s \end{pmatrix} & & \uparrow \begin{pmatrix} i_\eta \\ 2^s \end{pmatrix} & & \uparrow \begin{pmatrix} i_\eta \\ 2^s \end{pmatrix} \\ [C_\eta^7, S^5] & \xrightarrow{i_\eta^*} & [S^5, M_{2^r}^4 \vee S^5] & \xrightarrow{(2^{s'})^*} & [S^5, M_{2^r}^4 \vee S^5] \end{array}$$

which induces the following commutative diagram by (12)

$$\begin{array}{ccccc} [C_\eta^7, C_r^{6,s}] & \xrightarrow{i_\eta^*} & [S^5, C_r^{6,s}] & \xrightarrow{(2^{s'})^*} & [S^5, C_r^{6,s}] \\ \cong \uparrow & & \cong \uparrow & & \cong \uparrow \\ \frac{\mathbb{Z}/4 \oplus \mathbb{Z}}{\langle (2, 2^s) \rangle} & \xrightarrow{\varphi} & \frac{\mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (1, 2^s) \rangle} & \xrightarrow{(2^{s'})^*} & \frac{\mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (1, 2^s) \rangle} \end{array}$$

where $\varphi(1, 0) = 0$ and $\varphi(0, 1) = (0, 2)$. Under the isomorphisms

$$\frac{\mathbb{Z}/4 \oplus \mathbb{Z}}{\langle (2, 2^s) \rangle} \cong \mathbb{Z}/2 \langle (1, 2^{s-1}) \rangle \oplus \mathbb{Z}/2^{s+1} \langle (0, 1) \rangle; \quad \frac{\mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (1, 2^s) \rangle} \cong \mathbb{Z}/2^{s+1} \langle (0, 1) \rangle$$

we have $\mathbb{Z}/2 \oplus \mathbb{Z}/2^{s+1} \xrightarrow{(2^{s'})^* \varphi = \begin{pmatrix} 0 \\ 2^{s'+1} \end{pmatrix}} \mathbb{Z}/2^{s+1}$. Consequently,

$$\ker(i_\eta 2^{s'})^* \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}.$$

Thus there is a short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow [C_{s'}^7, C_r^{6,s}] \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1} \rightarrow 0$$

Together with (40), we have

$$\pi_9(C^{5,s'} \wedge C_r^{5,s}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}, \quad \text{i.e., } \hat{t} = 1.$$

From the analysis above, we get $\hat{\delta} = \delta$, $\hat{\alpha} = \alpha$ and $\hat{\gamma} = \gamma$, i.e., $Q_1 \simeq C^{5,s'} \wedge C_r^{5,s}$. \square

Hence $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_r^{9,s} \vee C^{5,s'} \wedge C_r^{5,s}$ for $s = r' > s' > r$.

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